Class: B. Tech (Unit III)

I have taken all course materials for Unit III from Book Concept of Modern Physics by Arthur Besier, Shobhit Mahajan & S. Rai Choudhury (McGraw Hill Education).

Students can download this book form given web address;

Web Address : **https://b-ok.cc/book/2700591/864ac0**

All topics of unit III (Special Theory of Relativity) have been taken from **Chapter1** from above said book (**https://b-ok.cc/book/2700591/864ac0**). I am sending pdf file of Chapter 3.

UNIT-III: Special Theory of Relativity **(8 Hours)**

Inertial Frames Of Reference, Galilian And Lorentz Transformations, Postulates of Relativity, Time Dilation, Twin Paradox, Length Contraction, Relativistic Mass, Energy and Momentum, Equivalence of Mass And Energy, Doppler Effect In Light And Its Application in Expanding of Universe, Problems.

Concepts of Modern Physics

Sixth Edition

CHAPTER 1

Relativity

According to the theory of relativity, nothing can travel faster than light. Although today's spacecraft can exceed 10 km/s, they are far from this ultimate speed limit.

In 1905 a young physicist of twenty-six named Albert Einstein showed how measurements of time and space are affected by motion between an observer and what is being observed. To say that Einstein's theory of relativity revolutionized science is no exaggeration. Relativity connects space and time, matter and energy, electricity and magnetism—links that are crucial to our understanding of the physical universe. From relativity have come a host of remarkable predictions, all of which have been confirmed by experiment. For all their profundity, many of the conclusions of relativity can be reached with only the simplest of mathematics.

1.1 SPECIAL RELATIVITY

All motion is relative; the speed of light in free space is the same for all observers

When such quantities as length, time interval, and mass are considered in elementary physics, no special point is made about how they are measured. Since a standard unit exists for each quantity, who makes a certain determination would not seem to matter—everybody ought to get the same result. For instance, there is no question of principle involved in finding the length of an airplane when we are on board. All we have to do is put one end of a tape measure at the airplane's nose and look at the number on the tape at the airplane's tail.

But what if the airplane is in flight and we are on the ground? It is not hard to determine the length of a distant object with a tape measure to establish a baseline, a surveyor's transit to measure angles, and a knowledge of trigonometry. When we measure the moving airplane from the ground, though, we find it to be shorter than it is to somebody in the airplane itself. To understand how this unexpected difference arises we must analyze the process of measurement when motion is involved.

Frames of Reference

The first step is to clarify what we mean by motion. When we say that something is moving, what we mean is that its position relative to something else is changing. A passenger moves relative to an airplane; the airplane moves relative to the earth; the earth moves relative to the sun; the sun moves relative to the galaxy of stars (the Milky Way) of which it is a member; and so on. In each case a **frame of reference** is part of the description of the motion. To say that something is moving always implies a specific frame of reference.

An **inertial frame of reference** is one in which Newton's first law of motion holds. In such a frame, an object at rest remains at rest and an object in motion continues to move at constant velocity (constant speed and direction) if no force acts on it. Any frame of reference that moves at constant velocity relative to an inertial frame is itself an inertial frame.

All inertial frames are equally valid. Suppose we see something changing its position with respect to us at constant velocity. Is it moving or are we moving? Suppose we are in a closed laboratory in which Newton's first law holds. Is the laboratory moving or is it at rest? These questions are meaningless because all constant-velocity motion is relative. There is no universal frame of reference that can be used everywhere, no such thing as "absolute motion."

The **theory of relativity** deals with the consequences of the lack of a universal frame of reference. **Special relativity,** which is what Einstein published in 1905, treats problems that involve inertial frames of reference. **General relativity**, published by Einstein a decade later, describes the relationship between gravity and the geometrical structure of space and time. The special theory has had an enormous impact on much of physics, and we shall concentrate on it here.

Postulates of Special Relativity

Two postulates underlie special relativity. The first, the **principle of relativity**, states:

> The laws of physics are the same in all inertial frames of reference.

This postulate follows from the absence of a universal frame of reference. If the laws of physics were different for different observers in relative motion, the observers could find from these differences which of them were "stationary" in space and which were "moving." But such a distinction does not exist, and the principle of relativity expresses this fact.

The second postulate is based on the results of many experiments:

> The speed of light in free space has the same value in all inertial frames of reference.

This speed is 2.998×10^8 m/s to four significant figures.

To appreciate how remarkable these postulates are, let us look at a hypothetical experiment basically no different from actual ones that have been carried out in a number of ways. Suppose I turn on a searchlight just as you fly past in a spacecraft at a speed of 2×10^8 m/s (Fig. 1.1). We both measure the speed of the light waves from the searchlight using identical instruments. From the ground I find their speed to be 3×10^8 m/s as usual. "Common sense" tells me that you ought to find a speed of $(3 - 2) \times 10^8$ m/s, or only 1×10^8 m/s, for the same light waves. But you also find their speed to be 3×10^8 m/s, even though to me you seem to be moving parallel to the waves at 2×10^8 m/s.

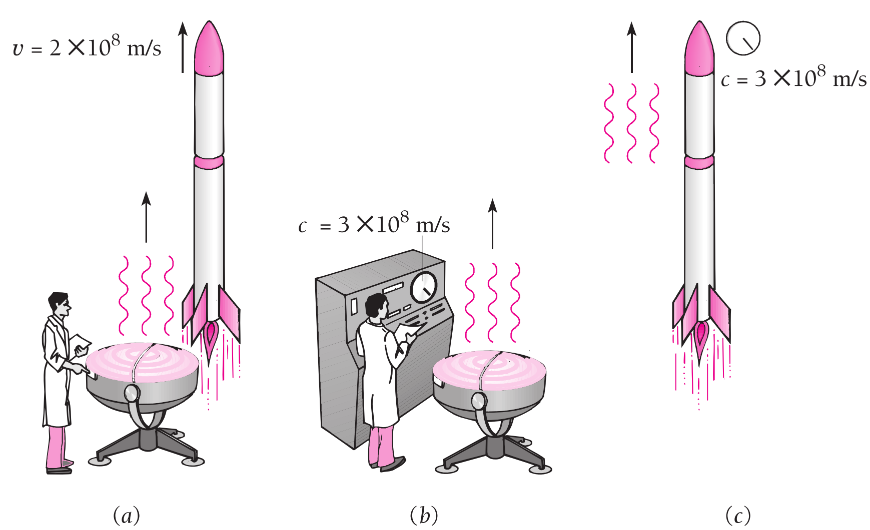


Figure 1.1 The speed of light is the same to all observers.

Albert A. Michelson (1852–1931) was born in Germany but came to the United States at the age of two with his parents, who settled in Nevada. He attended the U.S. Naval Academy at Annapolis where, after two years of sea duty, he became a science instructor. To improve his knowledge of optics, in which he wanted to specialize, Michelson went to Europe and studied in Berlin and Paris. Then he left the Navy to work first at the Case School of Applied Science in Ohio, then at Clark University in Massachusetts, and finally at the University of Chicago, where he headed the physics department from 1892 to 1929. Michelson's speciality was high-precision measurement, and for many decades his successive figures for the speed of light were the best available. He redefined the meter in terms of wavelengths of a particular spectral line and devised an interferometer that could determine the diameter of a star (stars appear as points of light in even the most powerful telescopes).

Michelson's most significant achievement, carried out in 1887 in collaboration with Edward Morley, was an experiment to measure the motion of the earth through the "ether," a hypothetical medium pervading the universe in which light waves were supposed to occur. The notion of the ether was a hangover from the days before light waves were recognized as electromagnetic, but nobody at the time seemed willing to discard the idea that light propagates relative to some sort of universal frame of reference.

To look for the earth's motion through the ether, Michelson and Morley used a pair of light beams formed by a half-silvered mirror, as in Fig. 1.2. One light beam is directed to a mirror along a path perpendicular to the ether current, and the other goes to a mirror along a path parallel to the ether current. Both beams end up at the same viewing screen. The clear glass plate ensures that both beams pass through the same thicknesses of air and glass. If the transit times of the two beams are the same, they will arrive at the screen in phase and will interfere constructively. An ether current due to the earth's motion parallel to one of the beams, however, would cause the beams to have different transit times and the result would be destructive interference at the screen. This is the essence of the experiment.

Although the experiment was sensitive enoug Y Y ø expected ether drift, to everyone's surprise none was found. The negative result had two consequences. First, it showed that the ether does not exist and so there is no such thing as "absolute motion" relative to the ether: all motion is relative to a specified frame of reference, not to a universal one. Second, the result showed that the speed of light is the same for all observers, which is not true of waves that need a material medium in which to occur (such as sound and water waves).

The Michelson-Morley experiment set the stage for Einstein's 1905 special theory of relativity, a theory that Michelson himself was reluctant to accept. Indeed, not long before the flowering of relativity and quantum theory revolutionized physics, Michelson announced that "physical discoveries in the future are a matter of the sixth decimal place." This was a common opinion of the time. Michelson received a Nobel Prize in 1907, the first American to do so.

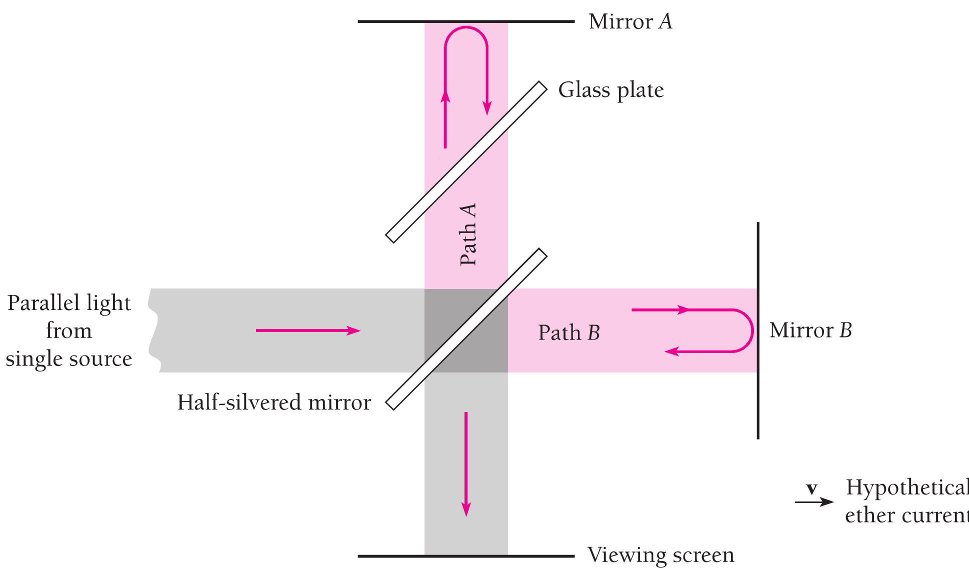


Figure 1.2 The Michelson-Morley experiment.

There is only one way to account for these results without violating the principle of relativity. It must be true that measurements of space and time are not absolute but depend on the relative motion between an observer and what is being observed. If I were to measure from the ground the rate at which your clock ticks and the length of your meter stick, I would find that the clock ticks more slowly than it did at rest on the ground and that the meter stick is shorter in the direction of motion of the spacecraft. To you, your clock and meter stick are the same as they were on the ground before you took off. To me they are different because of the relative motion, different in such a way that the speed of light you measure is the same 3×10^8 m/s I measure. Time intervals and lengths are relative quantities, but the speed of light in free space is the same to all observers.

Before Einstein's work, a conflict had existed between the principles of mechanics, which were then based on Newton's laws of motion, and those of electricity and magnetism, which had been developed into a unified theory by Maxwell. Newtonian mechanics had worked well for over two centuries. Maxwell's theory not only covered all that was then known about electric and magnetic phenomena but had also predicted that electromagnetic waves exist and identified light as an example of them. However, the equations of Newtonian mechanics and those of electromagnetism differ in the way they relate measurements made in one inertial frame with those made in a different inertial frame.

Einstein showed that Maxwell's theory is consistent with special relativity whereas Newtonian mechanics is not, and his modification of mechanics brought these branches of physics into accord. As we will find, relativistic and Newtonian mechanics agree for relative speeds much lower than the speed of light, which is why Newtonian mechanics seemed correct for so long. At higher speeds Newtonian mechanics fails and must be replaced by the relativistic version.

1.2 TIME DILATION

A moving clock ticks more slowly than a clock at rest

Measurements of time intervals are affected by relative motion between an observer and what is observed. As a result, a clock that moves with respect to an observer ticks more slowly than it does without such motion, and all processes (including those of life) occur more slowly to an observer when they take place in a different inertial frame.

If someone in a moving spacecraft finds that the time interval between two events in the spacecraft is t_0, we on the ground would find that the same interval has the longer duration t. The quantity t_0, which is determined by events that occur *at the same place* in an observer's frame of reference, is called the **proper time** of the interval between the events. When witnessed from the ground, the events that mark the beginning and end of the time interval occur at different places, and in consequence the duration of the interval appears longer than the proper time. This effect is called **time dilation** (to dilate is to become larger).

To see how time dilation comes about, let us consider two clocks, both of the particularly simple kind shown in Fig. 1.3. In each clock a pulse of light is reflected back and forth between two mirrors L_0 apart. Whenever the light strikes the lower mirror, an electric signal is produced that marks the recording tape. Each mark corresponds to the tick of an ordinary clock.

One clock is at rest in a laboratory on the ground and the other is in a spacecraft that moves at the speed $\boldsymbol{v}$ relative to the ground. An observer in the laboratory watches both clocks: does she find that they tick at the same rate?

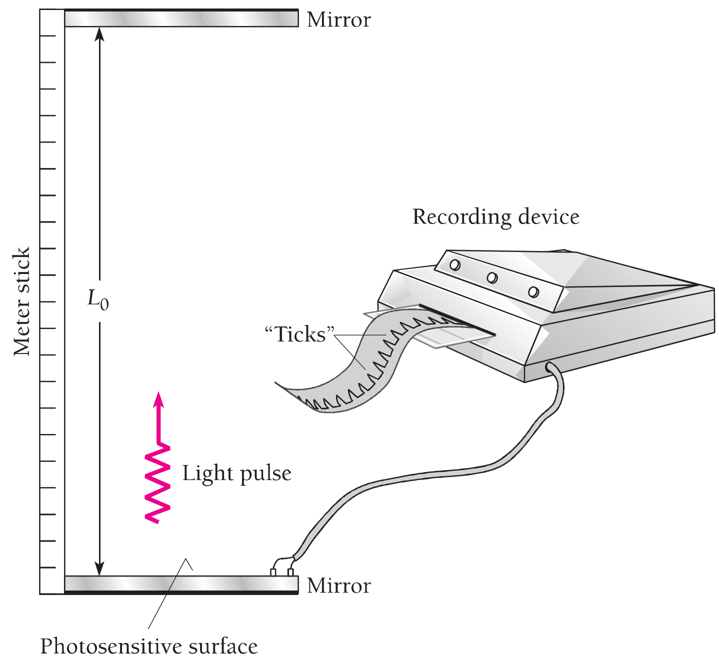


Figure 1.3 A simple clock. Each "tick" corresponds to a round trip of the light pulse from the lower mirror to the upper one and back.

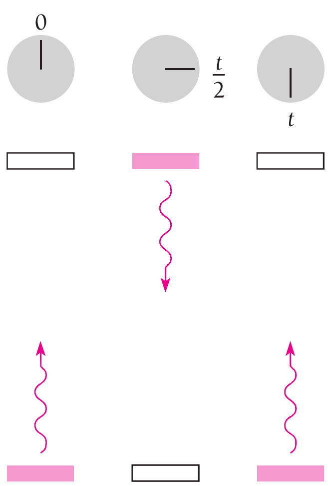


Figure 1.4 A light-pulse clock at rest on the ground as seen by an observer on the ground. The dial represents a conventional clock on the ground.

Figure 1.4 shows the laboratory clock in operation. The time interval between ticks is the proper time t_0 and the time needed for the light pulse to travel between the mirrors at the speed of light c is $t_0/2$. Hence $t_0/2 = L_0/c$ and

$$t_0 = \frac{2L_0}{c} \tag{1.1}$$

Figure 1.5 shows the moving clock with its mirrors perpendicular to the direction of motion relative to the ground. The time interval between ticks is t. Because the clock is moving, the light pulse, as seen from the ground, follows a zigzag path. On its way from the lower mirror to the upper one in the time $t/2$, the pulse travels a horizontal distance of $\boldsymbol{v}(t/2)$ and a total distance of $c(t/2)$. Since L_0 is the vertical distance between the mirrors,

$$\left(\frac{ct}{2}\right)^2 = L_0^2 + \left(\frac{\boldsymbol{v}t}{2}\right)^2$$

$$\frac{t^2}{4}(c^2 - \boldsymbol{v}^2) = L_0^2$$

$$t^2 = \frac{4L_0^2}{c^2 - \boldsymbol{v}^2} = \frac{(2L_0)^2}{c^2(1 - \boldsymbol{v}^2/c^2)}$$

$$t = \frac{2L_0/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \tag{1.2}$$

But $2L_0/c$ is the time interval t_0 between ticks on the clock on the ground, as in Eq. (1.1), and so

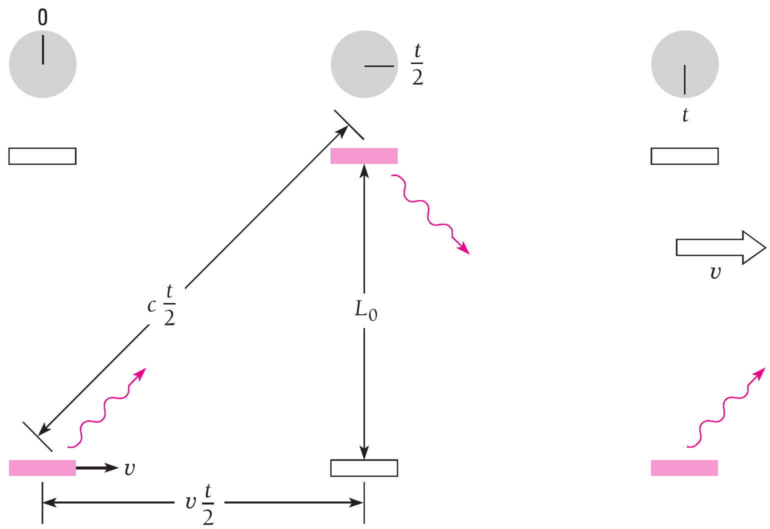


Figure 1.5 A light-pulse clock in a spacecraft as seen by an observer on the ground. The mirrors are parallel to the direction of motion of the spacecraft. The dial represents a conventional clock on the ground.

Time dilation

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}} \tag{1.3}$$

Here is a reminder of what the symbols in Eq. (1.4) represent:

t_0 = time interval on clock at rest relative to an observer = proper time
t = time interval on clock in motion relative to an observer
v = speed of relative motion
c = speed of light

Because the quantity $\sqrt{1 - v^2/c^2}$ is always smaller than 1 for a moving object, t is always greater than t_0. The moving clock in the spacecraft appears to tick at a slower rate than the stationary one on the ground, as seen by an observer on the ground.

Exactly the same analysis holds for measurements of the clock on the ground by the pilot of the spacecraft. To him, the light pulse of the ground clock follows a zigzag path that requires a total time t per round trip. His own clock, at rest in the spacecraft, ticks at intervals of t_0. He too finds that

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}}$$

so the effect is reciprocal: *every* observer finds that clocks in motion relative to him tick more slowly than clocks at rest relative to him.

Our discussion has been based on a somewhat unusual clock. Do the same conclusions apply to ordinary clocks that use machinery—spring-controlled escapements, tuning forks, vibrating quartz crystals, or whatever—to produce ticks at constant time intervals? The answer must be yes, since if a mirror clock and a conventional clock in the spacecraft agree with each other on the ground but not when in flight, the disagreement between then could be used to find the speed of the spacecraft independently of any outside frame of reference—which contradicts the principle that all motion is relative.

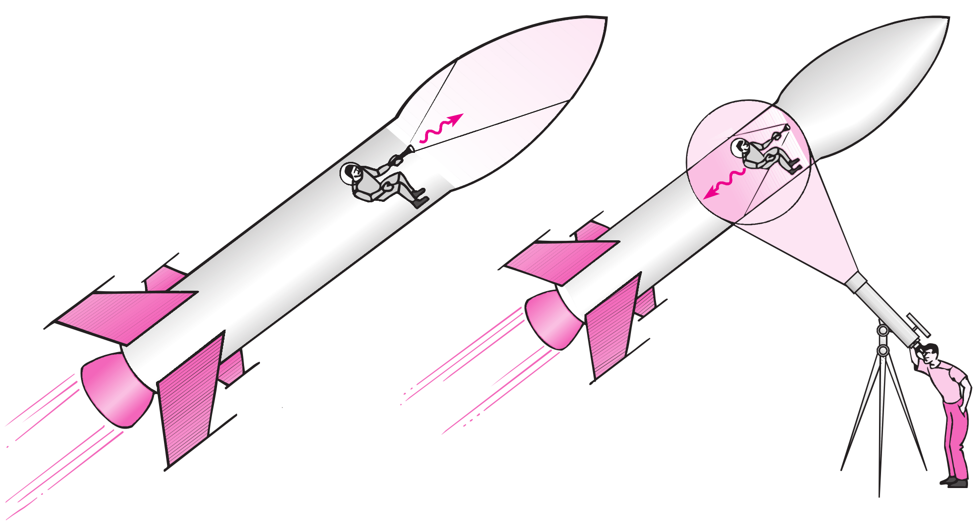
The Ultimate Speed Limit

The earth and the other planets of the solar system seem to be natural products of the evolution of the sun. Since the sun is a rather ordinary star in other ways, it is not surprising that other stars have been found to have planetary systems around them as well. Life developed here on earth, and there is no known reason why it should not also have done so on some of these planets. Can we expect ever to be able to visit them and meet our fellow citizens of the universe? The trouble is that nearly all stars are very far away—thousands or millions of light-years away. (A light-year, the distance light travels in a year, is 9.46×10^{15} m.) But if we can build a spacecraft whose speed is thousands or millions of times greater than the speed of light c, such distances would not be an obstacle.

Alas, a simple argument based on Einstein's postulates shows that nothing can move faster than c. Suppose you are in a spacecraft traveling at a constant speed $\mathbf{v}$ relative to the earth that is greater than c. As I watch from the earth, the lamps in the spacecraft suddenly go out. You switch on a flashlight to find the fuse box at the front of the spacecraft and change the blown fuse (Fig. 1.6*a*). The lamps go on again.

From the ground, though, I would see something quite different. To me, since your speed $\mathbf{v}$ is greater than c, the light from your flashlight illuminates the *back* of the spacecraft (Fig. 1.6*b*). I can only conclude that the laws of physics are different in your inertial frame from what they are in my inertial frame—which contradicts the principle of relativity. The only way to avoid this contradiction is to assume that nothing can move faster than the speed of light. This assumption has been tested experimentally many times and has always been found to be correct.

The speed of light c in relativity is always its value in free space of 3.00×10^8 m/s. In all material media, such as air, water, or glass, light travels more slowly than this, and atomic particles are able to move faster in such media than does light. When an electrically charged particle moves through a transparent substance at a speed exceeding that of light in the substance, a cone of light waves is emitted that corresponds to the bow wave produced by a ship moving through the water faster than water waves do. These light waves are known as **Cerenkov radiation** and form the basis of a method of determining the speeds of such particles. The minimum speed a particle must have to emit Cerenkov radiation is c/n in a medium whose index of refraction is n. Cerenkov radiation is visible as a bluish glow when an intense beam of particles is involved.



(*a*) (*b*)

Figure 1.6 A person switches on a flashlight in a spacecraft assumed to be moving relative to the earth faster than light. (*a*) In the spacecraft frame, the light goes to the front of the spacecraft. (*b*) In the earth frame, the light goes to the back of the spacecraft. Because observers in the spacecraft and on the earth would see different events, the principle of relativity would be violated. The conclusion is that the spacecraft cannot be moving faster than light relative to the earth (or relative to anything else).

(AIP Niels Bohr Library)

Albert Einstein (1879–1955), bitterly unhappy with the rigid discipline of the schools of his native Germany, went at sixteen to Switzerland to complete his education, and later got a job examining patent applications at the Swiss Patent Office. Then, in 1905, ideas that had been germinating in his mind for years when he should have been paying attention to other matters (one of his math teachers called Einstein a "lazy dog") blossomed into three short papers that were to change decisively the course not only of physics but of modern civilization as well.

The first paper, on the photoelectric effect, proposed that light has a dual character with both particle and wave properties. The subject of the second paper was Brownian motion, the irregular zigzag movement of tiny bits of suspended matter, such as pollen grains in water. Einstein showed that Brownian motion results from the bombardment of the particles by randomly moving molecules in the fluid in which they are suspended. This provided the long-awaited definite link with experiment that convinced the remaining doubters of the molecular theory of matter. The third paper introduced the special theory of relativity.

Although much of the world of physics was originally either indifferent or skeptical, even the most unexpected of Einstein's conclusions were soon confirmed and the development of what is now called modern physics began in earnest. After university posts in Switzerland and Czechoslovakia, in 1913 he took up an appointment at the Kaiser Wilhelm Institute in Berlin that left him able to do research free of financial worries and routine duties. Einstein's interest was now mainly in gravitation, and he started where Newton had left off more than two centuries earlier.

Einstein's general theory of relativity, published in 1916, related gravity to the structure of space and time. In this theory the force of gravity can be thought of as arising from a warping of spacetime around a body of matter so that a nearby mass tends to move toward it, much as a marble rolls toward the bottom of a saucer-shaped hole. From general relativity came a number of remarkable predictions, such as that light should be subject to gravity, all of which were verified experimentally. The later discovery that the universe is expanding fit neatly into the theory. In 1917 Einstein introduced the idea of stimulated emission of radiation, an idea that bore fruit forty years later in the invention of the laser.

The development of quantum mechanics in the 1920s disturbed Einstein, who never accepted its probabilistic rather than deterministic view of events on an atomic scale. "God does not play dice with the world," he said, but for once his physical intuition seemed to be leading him in the wrong direction.

Einstein, by now a world celebrity, left Germany in 1933 after Hitler came to power and spent the rest of his life at the Institute for Advanced Study in Princeton, New Jersey, thereby escaping the fate of millions of other European Jews at the hands of the Germans. His last years were spent in an unsuccessful search for a theory that would bring gravitation and electromagnetism together into a single picture, a problem worthy of his gifts but one that remains unsolved to this day.

Example 1.1

A spacecraft is moving relative to the earth. An observer on the earth finds that, between 1 P.M. and 2 P.M. according to her clock, 3601 s elapse on the spacecraft's clock. What is the spacecraft's speed relative to the earth?

Solution

Here $t_0 = 3600$ s is the proper time interval on the earth and $t = 3601$ s is the time interval in the moving frame as measured from the earth. We proceed as follows:

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}}$$

$$1 - \frac{v^2}{c^2} = \left(\frac{t_0}{t}\right)^2$$

$$v = c\sqrt{1 - \left(\frac{t_0}{t}\right)^2} = (2.998 \times 10^8 \text{ m/s})\sqrt{1 - \left(\frac{3600 \text{ s}}{3601 \text{ s}}\right)^2}$$

$$= 7.1 \times 10^6 \text{ m/s}$$

Today's spacecraft are much slower than this. For instance, the highest speed of the Apollo 11 spacecraft that went to the moon was only 10,840 m/s, and its clocks differed from those on the earth by less than one part in 10^9. Most of the experiments that have confirmed time dilation made use of unstable nuclei and elementary particles which readily attain speeds not far from that of light.

Apollo 11 lifts off its pad to begin the first human visit to the moon. At its highest speed of 10.8 km/s relative to the earth, its clocks differed from those on the earth by less than one part in a billion.

Although time is a relative quantity, not all the notions of time formed by everyday experience are incorrect. Time does not run backward to *any* observer, for instance. A sequence of events that occur at some particular point at $t_1, t_2, t_3, \ldots$ will appear in the same order to all observers everywhere, though not necessarily with the same time intervals $t_2 - t_1, t_3 - t_2, \ldots$ between each pair of events. Similarly, no distant observer, regardless of his or her state of motion, can see an event before it happens—more precisely, before a nearby observer sees it—since the speed of light is finite and signals require the minimum period of time L/c to travel a distance L. There is no way to peer into the future, although past events may appear different to different observers.

1.3 DOPPLER EFFECT

Why the universe is believed to be expanding

We are all familiar with the increase in pitch of a sound when its source approaches us (or we approach the source) and the decrease in pitch when the source recedes from us (or we recede from the source). These changes in frequency constitute the **doppler effect,** whose origin is straightforward. For instance, successive waves emitted by a source moving toward an observer are closer together than normal because of the advance of the source; because the separation of the waves is the wavelength of the sound, the corresponding frequency is higher. The relationship between the source frequency ν_0 and the observed frequency ν is

Doppler effect in sound

$$\nu = \nu_0\left(\frac{1 + v/c}{1 - V/c}\right) \tag{1.4}$$

where c = speed of sound
v = speed of observer (+ for motion toward the source, − for motion away from it)
V = speed of the source (+ for motion toward the observer, − for motion away from him)

If the observer is stationary, $v = 0$, and if the source is stationary, $V = 0$.

The doppler effect in sound varies depending on whether the source, or the observer, or both are moving. This appears to violate the principle of relativity: all that should count is the relative motion of source and observer. But sound waves occur only in a material medium such as air or water, and this medium is itself a frame of reference with respect to which motions of source and observer are measurable. Hence there is no contradiction. In the case of light, however, no medium is involved and only relative motion of source and observer is meaningful. The doppler effect in light must therefore differ from that in sound.

We can analyze the doppler effect in light by considering a light source as a clock that ticks ν_0 times per second and emits a wave of light with each tick. We will examine the three situations shown in Fig. 1.7.

1 *Observer moving perpendicular to a line between him and the light source*. The proper time between ticks is $t_0 = 1/\nu_0$, so between one tick and the next the time $t = t_0/\sqrt{1 - v^2/c^2}$ elapses in the reference frame of the observer. The frequency he finds is accordingly

$$\nu(\text{transverse}) = \frac{1}{t} = \frac{\sqrt{1 - v^2/c^2}}{t_0}$$

Transverse doppler effect in light

$$\nu = \nu_0\sqrt{1 - v^2/c^2} \tag{1.5}$$

The observed frequency ν is always lower than the source frequency ν_0.

2 *Observer receding from the light source*. Now the observer travels the distance vt away from the source between ticks, which means that the light wave from a given tick takes

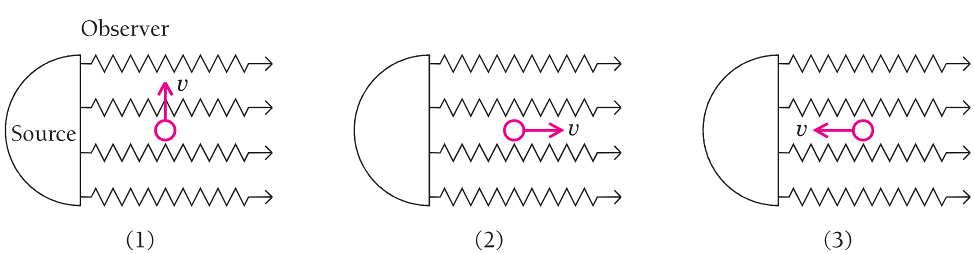


Figure 1.7 The frequency of the light seen by an observer depends on the direction and speed of the observer's motion relative to its source.

vt/c longer to reach him than the previous one. Hence the total time between the arrival of successive waves is

$$T = t + \frac{vt}{c} = t_0 \frac{1 + v/c}{\sqrt{1 - v^2/c^2}} = t_0 \frac{\sqrt{1 + v/c}\sqrt{1 + v/c}}{\sqrt{1 + v/c}\sqrt{1 - v/c}} = t_0 \sqrt{\frac{1 + v/c}{1 - v/c}}$$

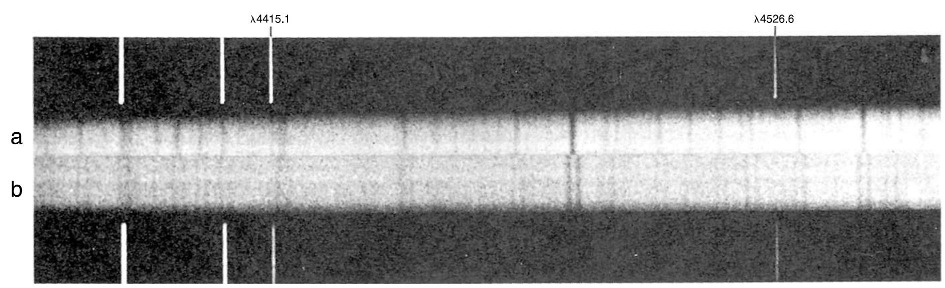
and the observed frequency is

$$\nu(\text{receding}) = \frac{1}{T} = \frac{1}{t_0}\sqrt{\frac{1 - v/c}{1 + v/c}} = \nu_0 \sqrt{\frac{1 - v/c}{1 + v/c}} \tag{1.6}$$

The observed frequency ν is lower than the source frequency ν_0. Unlike the case of sound waves, which propagate relative to a material medium it makes no difference whether the observer is moving away from the source or the source is moving away from the observer.

3 *Observer approaching the light source*. The observer here travels the distance vt toward the source between ticks, so each light wave takes vt/c less time to arrive than the previous one. In this case $T = t - vt/c$ and the result is

$$\nu(\text{approaching}) = \nu_0 \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{1.7}$$



Spectra of the double star Mizar, which consists of two stars that circle their center of mass, taken 2 days apart. In *a* the stars are in line with no motion toward or away from the earth, so their spectral lines are superimposed. In *b* one star is moving toward the earth and the other is moving away from the earth, so the spectral lines of the former are doppler-shifted toward the blue end of the spectrum and those of the latter are shifted toward the red end.

The observed frequency is higher than the source frequency. Again, the same formula holds for motion of the source toward the observer.

Equations (1.6) and (1.7) can be combined in the single formula

Longitudinal doppler effect in light

$$\nu = \nu_0 \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{1.8}$$

by adopting the convention that v is + for source and observer approaching each other and − for source and observer receding from each other.

Example 1.2

A driver is caught going through a red light. The driver claims to the judge that the color she actually saw was green ($\nu = 5.60 \times 10^{14}$ Hz) and not red ($\nu_0 = 4.80 \times 10^{14}$ Hz) because of the doppler effect. The judge accepts this explanation and instead fines her for speeding at the rate of \$1 for each km/h she exceeded the speed limit of 80 km/h. What was the fine?

Solution

Solving Eq. (1.8) for v gives

$$v = c\left(\frac{\nu^2 - \nu_0^2}{\nu^2 + \nu_0^2}\right) = (3.00 \times 10^8 \text{ m/s})\left[\frac{(5.60)^2 - (4.80)^2}{(5.60)^2 + (4.80)^2}\right]$$

$$= 4.59 \times 10^7 \text{ m/s} = 1.65 \times 10^8 \text{ km/h}$$

since 1 m/s = 3.6 km/h. The fine is therefore $\$(1.65 \times 10^8 - 80) = \$164,999,920$.

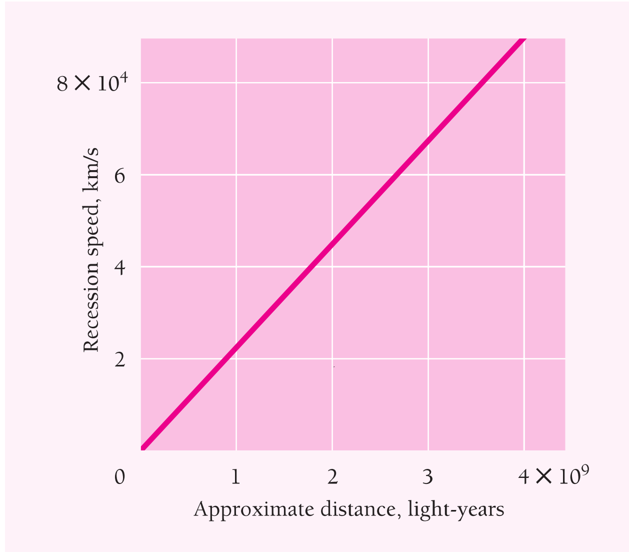
Visible light consists of electromagnetic waves in a frequency band to which the eye is sensitive. Other electromagnetic waves, such as those used in radar and in radio communications, also exhibit the doppler effect in accord with Eq. (1.8). Doppler shifts in radar waves are used by police to measure vehicle speeds, and doppler shifts in the radio waves emitted by a set of earth satellites formed the basis of the highly accurate Transit system of marine navigation.

The Expanding Universe

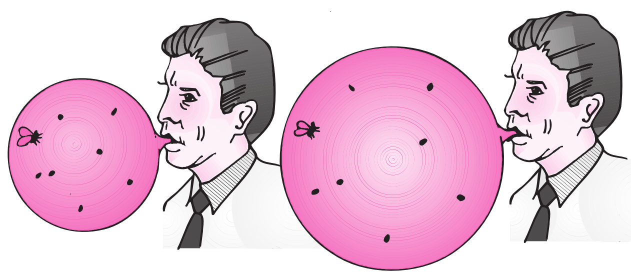
The doppler effect in light is an important tool in astronomy. Stars emit light of certain characteristic frequencies called spectral lines, and motion of a star toward or away from the earth shows up as a doppler shift in these frequencies. The spectral lines of distant galaxies of stars are all shifted toward the low-frequency (red) end of the spectrum and hence are called "red shifts." Such shifts indicate that the galaxies are receding from us and from one another. The speeds of recession are observed to be

Edwin Hubble (1889–1953) was born in Missouri and, although always interested in astronomy, pursued a variety of other subjects as well at the University of Chicago. He then went as a Rhodes Scholar to Oxford University in England where he concentrated on law, Spanish, and heavyweight boxing. After two years of teaching at an Indiana high school, Hubble realized what his true vocation was and returned to the University of Chicago to study astronomy.

At Mt. Wilson Observatory in California, Hubble made the first accurate measurements of the distances of spiral galaxies which showed that they are far away in space from our own Milky Way galaxy. It had been known for some time that such galaxies have red shifts in their spectra that indicate motion away from the Milky Way, and Hubble joined his distance figures with the observed red shifts to conclude that the recession speeds were proportional to distance. This implies that the universe is expanding, a remarkable discovery that has led to the modern picture of the universe. Hubble was the first to use the 200-inch telescope, for many years the world's largest, at Mt. Palomar in California, in 1949. In his later work Hubble tried to determine the structure of the universe by finding how the concentration of remote galaxies varies with distance, a very difficult task that only today is being accomplished.



(*a*)



(*b*)

Figure 1.8 (*a*) Graph of recession speed versus distance for distant galaxies. The speed of recession averages about 21 km/s per million light-years. (*b*) Two-dimensional analogy of the expanding universe. As the balloon is inflated, the spots on it become farther apart. A bug on the balloon would find that the farther away a spot is from its location, the faster the spot seems to be moving away; this is true no matter where the bug is. In the case of the universe, the more distant a galaxy is from us, the faster it is moving away, which means that the universe is expanding uniformly.

proportional to distance, which suggests that the entire universe is expanding (Fig. 1.8). This proportionality is called **Hubble's law.**

The expansion apparently began about 13 billion years ago when a very small, intensely hot mass of primeval matter exploded, an event usually called the **Big Bang.** As described in Chap. 13, the matter soon turned into the electrons, protons, and neutrons of which the present universe is composed. Individual aggregates that formed during the expansion became the galaxies of today. Present data suggest that the current expansion will continue forever.

Example 1.3

A distant galaxy in the constellation Hydra is receding from the earth at 6.12×10^7 m/s. By how much is a green spectral line of wavelength 500 nm (1 nm $= 10^{-9}$ m) emitted by this galaxy shifted toward the red end of the spectrum?

Solution

Since $\lambda = c/\nu$ and $\lambda_0 = c/\nu_0$, from Eq. (1.6) we have

$$\lambda = \lambda_0 \sqrt{\frac{1 + v/c}{1 - v/c}}$$

Here $v = 0.204c$ and $\lambda_0 = 500$ nm, so

$$\lambda = 500 \text{ nm} \sqrt{\frac{1 + 0.204}{1 - 0.204}} = 615 \text{ nm}$$

which is in the orange part of the spectrum. The shift is $\lambda - \lambda_0 = 115$ nm. This galaxy is believed to be 2.9 billion light-years away.

1.4 LENGTH CONTRACTION

Faster means shorter

Measurements of lengths as well as of time intervals are affected by relative motion. The length L of an object in motion with respect to an observer always appears to the observer to be shorter than its length L_0 when it is at rest with respect to him. This contraction occurs only in the direction of the relative motion. The length L_0 of an object in its rest frame is called its **proper length.** (We note that in Fig. 1.5 the clock is moving perpendicular to **v**, hence $L = L_0$ there.)

The length contraction can be derived in a number of ways. Perhaps the simplest is based on time dilation and the principle of relativity. Let us consider what happens to unstable particles called muons that are created at high altitudes by fast cosmic-ray particles (largely protons) from space when they collide with atomic nuclei in the earth's atmosphere. A muon has a mass 207 times that of the electron and has a charge of either $+e$ or $-e$; it decays into an electron or a positron after an average lifetime of 2.2 μs (2.2×10^{-6} s).

Cosmic-ray muons have speeds of about 2.994×10^8 m/s (0.998c) and reach sea level in profusion—one of them passes through each square centimeter of the earth's surface on the average slightly more often than once a minute. But in $t_0 = 2.2\ \mu$s, their average lifetime, muons can travel a distance of only

$$vt_0 = (2.994 \times 10^8 \text{ m/s})(2.2 \times 10^{-6} \text{ s}) = 6.6 \times 10^2 \text{ m} = 0.66 \text{ km}$$

before decaying, whereas they are actually created at altitudes of 6 km or more.

To resolve the paradox, we note that the muon lifetime of $t_0 = 2.2\ \mu$s is what an observer at rest with respect to a muon would find. Because the muons are hurtling toward us at the considerable speed of 0.998c, their lifetimes are extended in our frame of reference by time dilation to

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}} = \frac{2.2 \times 10^{-6} \text{ s}}{\sqrt{1 - (0.998c)^2/c^2}} = 34.8 \times 10^{-6} \text{ s} = 34.8\ \mu\text{s}$$

The moving muons have lifetimes almost 16 times longer than those at rest. In a time interval of 34.8 μs, a muon whose speed is 0.998c can cover the distance

$$vt = (2.994 \times 10^8 \text{ m/s})(34.8 \times 10^{-6} \text{ s}) = 1.04 \times 10^4 \text{ m} = 10.4 \text{ km}$$

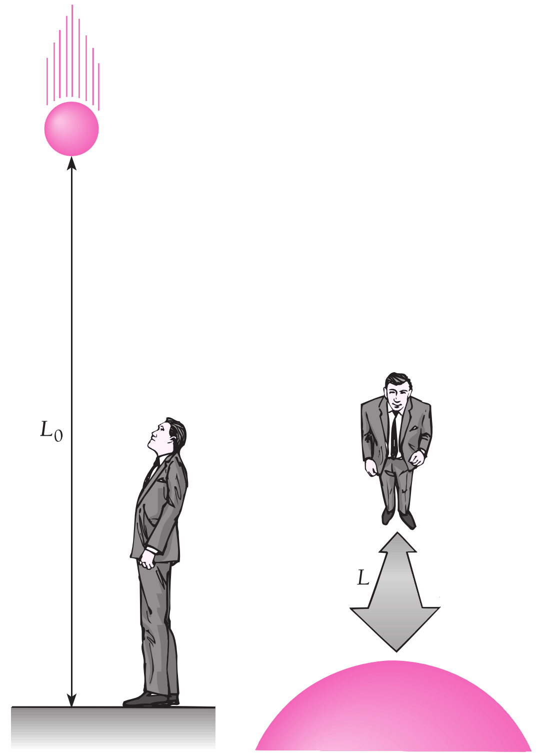


Figure 1.9 Muon decay as seen by different observers. The muon size is greatly exaggerated here; in fact, the muon seems likely to be a point particle with no extension in space.

Although its lifetime is only $t_0 = 2.2\ \mu$s in its own frame of reference, a muon can reach the ground from altitudes of as much as 10.4 km because in the frame in which these altitudes are measured, the muon lifetime is $t = 34.8\ \mu$s.

What if somebody were to accompany a muon in its descent at $v = 0.998c$, so that to him or her the muon is at rest? The observer and the muon are now in the same frame of reference, and in this frame the muon's lifetime is only 2.2 μs. To the observer, the muon can travel only 0.66 km before decaying. The only way to account for the arrival of the muon at ground level is if the distance it travels, from the point of view of an observer in the moving frame, is shortened by virtue of its motion (Fig. 1.9). The principle of relativity tells us the extent of the shortening—it must be by the same factor of $\sqrt{1 - v^2/c^2}$ that the muon lifetime is extended from the point of view of a stationary observer.

We therefore conclude that an altitude we on the ground find to be h_0 must appear in the muon's frame of reference as the lower altitude

$$h = h_0 \sqrt{1 - v^2/c^2}$$

In our frame of reference the muon can travel $h_0 = 10.4$ km because of time dilation. In the muon's frame of reference, where there is no time dilation, this distance is abbreviated to

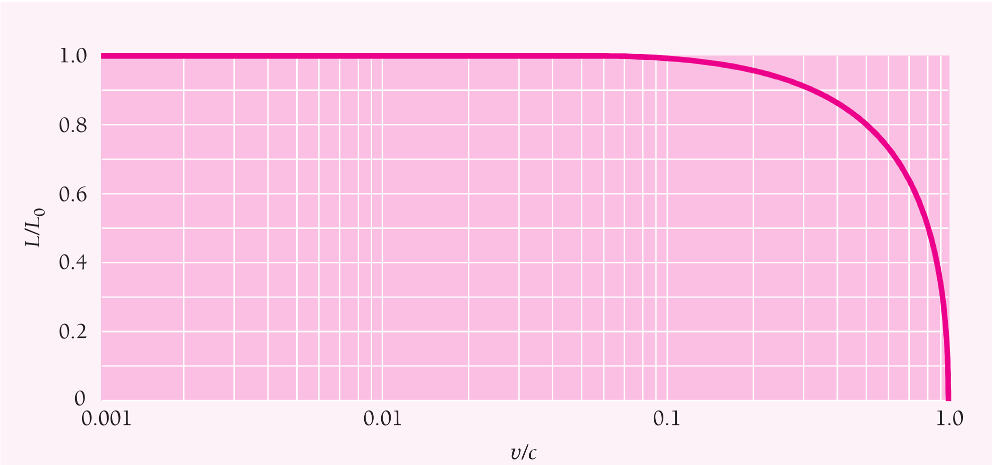


Figure 1.10 Relativistic length contraction. Only lengths in the direction of motion are affected. The horizontal scale is logarithmic.

$$h = (10.4 \text{ km}) \sqrt{1 - (0.998c)^2/c^2} = 0.66 \text{ km}$$

As we know, a muon traveling at $0.998c$ goes this far in 2.2 μs.

The relativistic shortening of distances is an example of the general contraction of lengths in the direction of motion:

Length contraction

$$L = L_0 \sqrt{1 - v^2/c^2} \tag{1.9}$$

Figure 1.10 is a graph of L/L_0 versus v/c. Clearly the length contraction is most significant at speeds near that of light. A speed of 1000 km/s seems fast to us, but it only results in a shortening in the direction of motion to 99.9994 percent of the proper length of an object moving at this speed. On the other hand, something traveling at nine-tenths the speed of light is shortened to 44 percent of its proper length, a significant change.

Like time dilation, the length contraction is a reciprocal effect. To a person in a spacecraft, objects on the earth appear shorter than they did when he or she was on the ground by the same factor of $\sqrt{1 - v^2/c^2}$ that the spacecraft appears shorter to somebody at rest. The proper length L_0 found in the rest frame is the maximum length any observer will measure. As mentioned earlier, only lengths in the direction of motion undergo contraction. Thus to an outside observer a spacecraft is shorter in flight than on the ground, but it is not narrower.

1.5 TWIN PARADOX

A longer life, but it will not seem longer

We are now in a position to understand the famous relativistic effect known as the twin paradox. This paradox involves two identical clocks, one of which remains on the earth while the other is taken on a voyage into space at the speed v and eventually is brought back. It is customary to replace the clocks with the pair of twins Dick and

Jane, a substitution that is perfectly acceptable because the processes of life—heartbeats, respiration, and so on—constitute biological clocks of reasonable regularity.

Dick is 20 y old when he takes off on a space voyage at a speed of 0.80c to a star 20 light-years away. To Jane, who stays behind, the pace of Dick's life is slower than hers by a factor of

$$\sqrt{1 - v^2/c^2} = \sqrt{1 - (0.80c)^2/c^2} = 0.60 = 60\%$$

To Jane, Dick's heart beats only 3 times for every 5 beats of her heart; Dick takes only 3 breaths for every 5 of hers; Dick thinks only 3 thoughts for every 5 of hers. Finally Dick returns after 50 years have gone by according to Jane's calendar, but to Dick the trip has taken only 30 y. Dick is therefore 50 y old whereas Jane, the twin who stayed home, is 70 y old (Fig. 1.11).

Where is the paradox? If we consider the situation from the point of view of Dick in the spacecraft, Jane on the earth is in motion relative to him at a speed of 0.80c. Should not Jane then be 50 y old when the spacecraft returns, while Dick is then 70—the precise opposite of what was concluded above?

But the two situations are not equivalent. Dick changed from one inertial frame to a different one when he started out, when he reversed direction to head home, and when he landed on the earth. Jane, however, remained in the same inertial frame during Dick's whole voyage. The time dilation formula applies to Jane's observations of Dick, but not to Dick's observations of her.

To look at Dick's voyage from his perspective, we must take into account that the distance L he covers is shortened to

$$L = L_0\sqrt{1 - v^2/c^2} = (20 \text{ light-years})\sqrt{1 - (0.80c)^2/c^2} = 12 \text{ light-years}$$

To Dick, time goes by at the usual rate, but his voyage to the star has taken $L/v = 15$ y and his return voyage another 15 y, for a total of 30 y. Of course, Dick's life span has

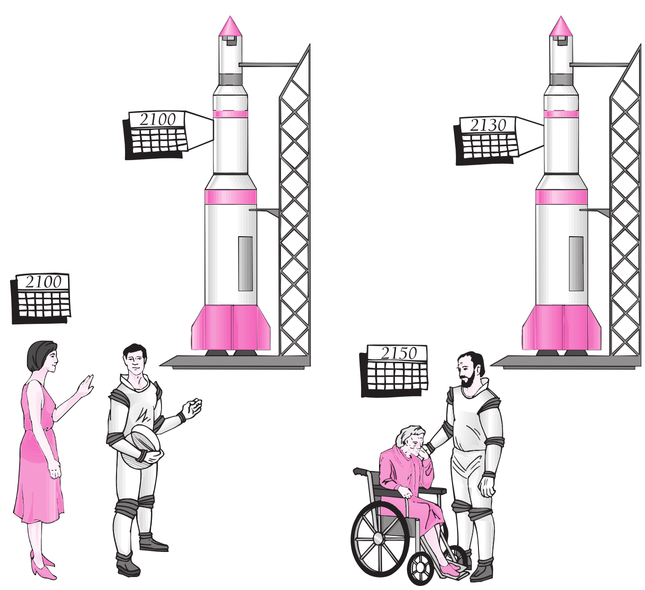


Figure 1.11 An astronaut who returns from a space voyage will be younger than his or her twin who remains on earth. Speeds close to the speed of light (here $v = 0.8c$) are needed for this effect to be conspicuous.

not been extended *to him*, because regardless of Jane's 50-y wait, he has spent only 30 y on the roundtrip.

The nonsymmetric aging of the twins has been verified by experiments in which accurate clocks were taken on an airplane trip around the world and then compared with identical clocks that had been left behind. An observer who departs from an inertial system and then returns after moving relative to that system will always find his or her clocks slow compared with clocks that stayed in the system.

Example 1.4

Dick and Jane each send out a radio signal once a year while Dick is away. How many signals does Dick receive? How many does Jane receive?

Solution

On the outward trip, Dick and Jane are being separated at a rate of $0.80c$. With the help of the reasoning used to analyze the doppler effect in Sec. 1.3, we find that each twin receives signals

$$T_1 = t_0 \sqrt{\frac{1 + v/c}{1 - v/c}} = (1 \text{ y}) \sqrt{\frac{1 + 0.80}{1 - 0.80}} = 3 \text{ y}$$

apart. On the return trip, Dick and Jane are getting closer together at the same rate, and each receives signals more frequently, namely

$$T_2 = t_0 \sqrt{\frac{1 - v/c}{1 + v/c}} = (1 \text{ y}) \sqrt{\frac{1 - 0.80}{1 + 0.80}} = \frac{1}{3} \text{ y}$$

apart.

To Dick, the trip to the star takes 15 y, and he receives $15/3 = 5$ signals from Jane. During the 15 y of the return trip, Dick receives $15/(1/3) = 45$ signals from Jane, for a total of 50 signals. Dick therefore concludes that Jane has aged by 50 y in his absence. Both Dick and Jane agree that Jane is 70 y old at the end of the voyage.

To Jane, Dick needs $L_0/v = 25$ y for the outward trip. Because the star is 20 light-years away. Jane on the earth continues to receive Dick's signals at the original rate of one every 3 y for 20 y after Dick has arrived at the star. Hence Jane receives signals every 3 y for 25 y + 20 y = 45 y to give a total of $45/3 = 15$ signals. (These are the 15 signals Dick sent out on the outward trip.) Then, for the remaining 5 y of what is to Jane a 50-y voyage, signals arrive from Dick at the shorter intervals of $1/3$ y for an additional $5/(1/3) = 15$ signals. Jane thus receives 30 signals in all and concludes that Dick has aged by 30 y during the time he was away—which agrees with Dick's own figure. Dick is indeed 20 y younger than his twin Jane on his return.

1.6 ELECTRICITY AND MAGNETISM

Relativity is the bridge

One of the puzzles that set Einstein on the trail of special relativity was the connection between electricity and magnetism, and the ability of his theory to clarify the nature of this connection is one of its triumphs.

Because the moving charges (usually electrons) whose interactions give rise to many of the magnetic forces familiar to us have speeds far smaller than c, it is not obvious that the operation of an electric motor, say, is based on a relativistic effect. The idea becomes less implausible, however, when we reflect on the strength of electric forces. The electric attraction between the electron and proton in a hydrogen atom, for instance, is 10^{39} times greater than the gravitational attraction between them. Thus even a small change in the character of these forces due to relative motion, which is what magnetic forces represent, may have large consequences. Furthermore, although the effective speed of an individual electron in a current-carrying wire (<1 mm/s) is less than that of a tired caterpillar, there may be 10^{20} or more moving electrons per centimeter in such a wire, so the total effect may be considerable.

Although the full story of how relativity links electricity and magnetism is mathematically complex, some aspects of it are easy to appreciate. An example is the origin of the magnetic force between two parallel currents. An important point is that, like the speed of light,

Electric charge is relativistically invariant.

A charge whose magnitude is found to be Q in one frame of reference is also Q in all other frames.

Let us look at the two idealized conductors shown in Fig. 1.12*a*. They contain equal numbers of positive and negative charges at rest that are equally spaced. Because the conductors are electrically neutral, there is no force between them.

Figure 1.12*b* shows the same conductors when they carry currents i_{I} and i_{II} in the same direction. The positive charges move to the right and the negative charges move to the left, both at the same speed $\boldsymbol{v}$ as seen from the laboratory frame of reference. (Actual currents in metals consist of flows of negative electrons only, of course, but the electrically equivalent model here is easier to analyze and the results are the same.) Because the charges are moving, their spacing is smaller than before by the factor $\sqrt{1 - v^2/c^2}$. Since $\boldsymbol{v}$ is the same for both sets of charges, their spacings shrink by the same amounts, and both conductors remain neutral to an observer in the laboratory. However, the conductors now attract each other. Why?

Let us look at conductor II from the frame of reference of one of the negative charges in conductor I. Because the negative charges in II appear at rest in this frame, their spacing is not contracted, as in Fig. 1.12*c*. On the other hand, the positive charges in II now have the velocity $2\boldsymbol{v}$, and their spacing is accordingly contracted to a greater extent than they are in the laboratory frame. Conductor II therefore appears to have a net positive charge, and an attractive force acts on the negative charge in I.

Next we look at conductor II from the frame of reference of one of the positive charges in conductor I. The positive charges in II are now at rest, and the negative charges there move to the left at the speed $2\boldsymbol{v}$. Hence the negative charges are closer together than the positive ones, as in Fig. 1.12*d*, and the entire conductor appears negatively charged. An attractive force therefore acts on the positive charges in I.

Identical arguments show that the negative and positive charges in II are attracted to I. Thus all the charges in each conductor experience forces directed toward the other conductor. To each charge, the force on it is an "ordinary" electric force that arises because the charges of opposite sign in the other conductor are closer together than the charges of the same sign, so the other conductor appears to have a net charge. From the laboratory frame the situation is less straightforward. Both conductors are electrically neutral in this frame, and it is natural to explain their mutual attraction by attributing it to a special "magnetic" interaction between the currents.

A similar analysis explains the repulsive force between parallel conductors that carry currents in opposite directions. Although it is convenient to think of magnetic forces as being different from electric ones, they both result from a single electromagnetic interaction that occurs between charged particles.

Clearly a current-carrying conductor that is electrically neutral in one frame of reference might not be neutral in another frame. How can this observation be reconciled

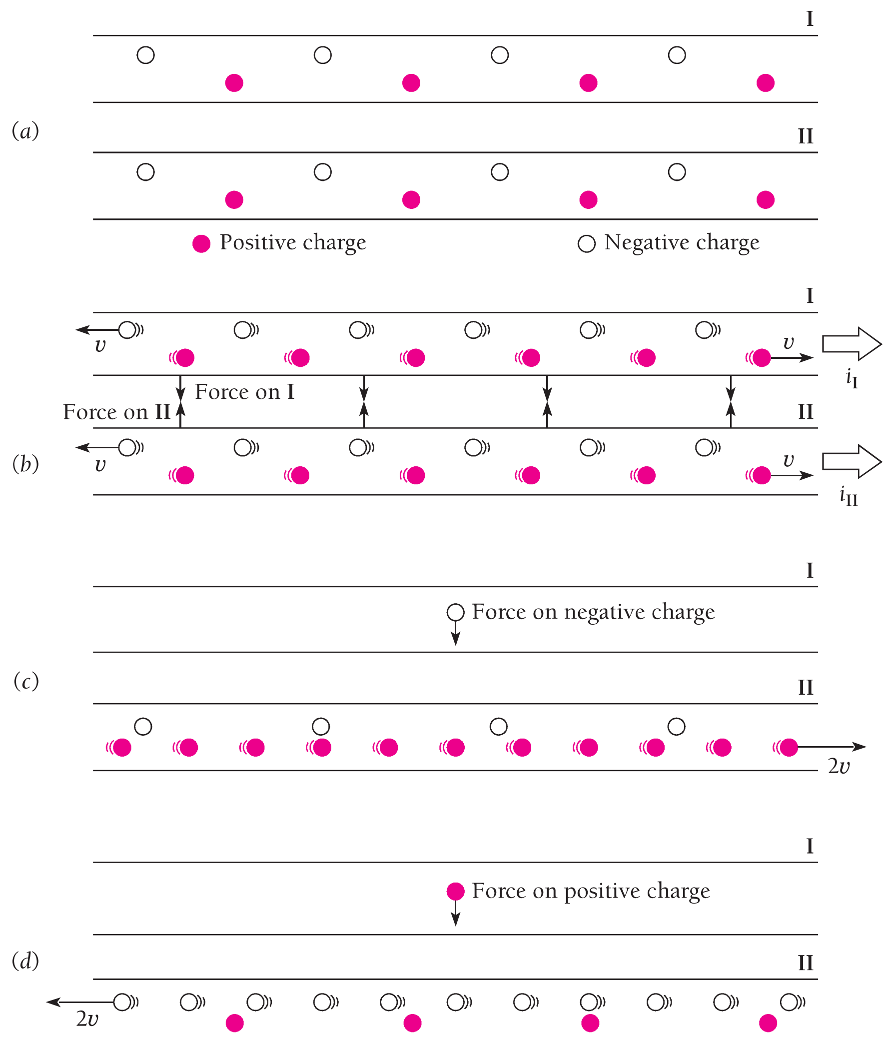


Figure 1.12 How the magnetic attraction between parallel currents arises. (*a*) Idealized parallel conductors that contain equal numbers of positive and negative charges. (*b*) When the conductors carry currents, the spacing of their moving charges undergoes a relativistic contraction as seen from the laboratory. The conductors attract each other when i_I and i_{II} are in the same direction. (*c*) As seen by a negative charge in I, the negative charges in II are at rest whereas the positive charges are in motion. The contracted spacing of the latter leads to a net positive charge in II that attracts the negative charge in I. (*d*) As seen by a positive charges in I, the positive charges in II are at rest whereas the negative charges are in motion. The contracted spacing of the latter leads to a net negative charge on II that attrats the positive charge in I. The contracted spacings in *b*, *c*, and *d* are greatly exaggerated.

with charge invariance? The answer is that we must consider the entire circuit of which the conductor is a part. Because the circuit must be closed for a current to occur in it, for every current element in one direction that a moving observer finds to have, say, a positive charge, there must be another current element in the opposite direction which the same observer finds to have a negative charge. Hence magnetic forces always act between different parts of the same circuit, even though the circuit as a whole appears electrically neutral to all observers.

The preceding discussion considered only a particular magnetic effect. All other magnetic phenomena can also be interpreted on the basis of Coulomb's law, charge invariance, and special relativity, although the analysis is usually more complicated.

1.7 RELATIVISTIC MOMENTUM

Redefining an important quantity

In classical mechanics linear momentum $\mathbf{p} = m\mathbf{v}$ is a useful quantity because it is conserved in a system of particles not acted upon by outside forces. When an event such as a collision or an explosion occurs inside an isolated system, the vector sum of the momenta of its particles before the event is equal to their vector sum afterward. We now have to ask whether $\mathbf{p} = m\mathbf{v}$ is valid as the definition of momentum in inertial frames in relative motion, and if not, what a relativistically correct definition is.

To start with, we require that $\mathbf{p}$ be conserved in a collision for all observers in relative motion at constant velocity. Also, we know that $\mathbf{p} = m\mathbf{v}$ holds in classical mechanics, that is, for $v \ll c$. Whatever the relativistically correct $\mathbf{p}$ is, then, it must reduce to $m\mathbf{v}$ for such velocities.

Let us consider an elastic collision (that is, a collision in which kinetic energy is conserved) between two particles A and B, as witnessed by observers in the reference frames S and S' which are in uniform relative motion. The properties of A and B are identical when determined in reference frames in which they are at rest. The frames S and S' are oriented as in Fig. 1.13, with S' moving in the $+x$ direction with respect to S at the velocity $\mathbf{v}$.

Before the collision, particle A had been at rest in frame S and particle B in frame S'. Then, at the same instant, A was thrown in the $+y$ direction at the speed V_A while B was thrown in the $-y'$ direction at the speed V'_B, where

$$V_A = V'_B \tag{1.10}$$

Hence the behavior of A as seen from S is exactly the same as the behavior of B as seen from S'.

When the two particles collide, A rebounds in the $-y$ direction at the speed V_A, while B rebounds in the $+y'$ direction at the speed V'_B. If the particles are thrown from positions Y apart, an observer in S finds that the collision occurs at $y = \frac{1}{2}Y$ and one in S' finds that it occurs at $y' = y = \frac{1}{2}Y$. The round-trip time T_0 for A as measured in frame S is therefore

$$T_0 = \frac{Y}{V_A} \tag{1.11}$$

and it is the same for B in S':

$$T_0 = \frac{Y}{V'_B}$$

In S the speed V_B is found from

$$V_B = \frac{Y}{T} \tag{1.12}$$

where T is the time required for B to make its round trip *as measured in S*. In S', however, B's trip requires the time T_0, where

$$T = \frac{T_0}{\sqrt{1 - v^2/c^2}} \tag{1.13}$$

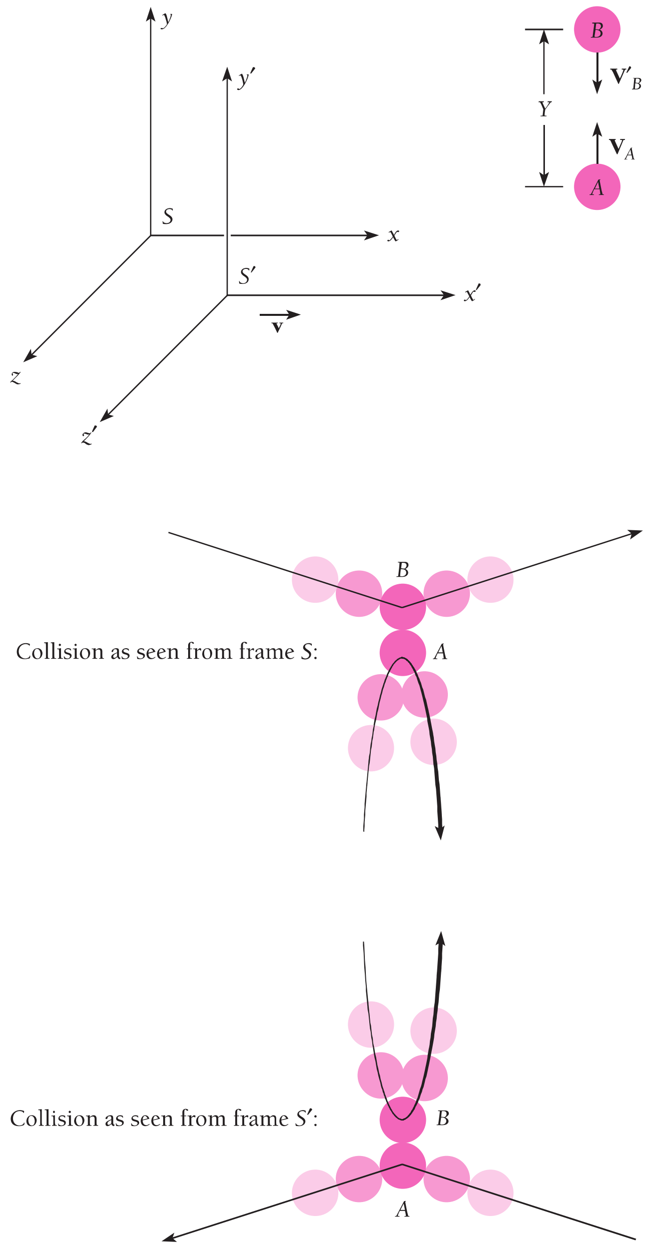


Figure 1.13 An elastic collision as observed in two different frames of reference. The balls are initially Y apart, which is the same distance in both frames since S' moves only in the x direction.

according to our previous results. Although observers in both frames see the same event, they disagree about the length of time the particle thrown from the other frame requires to make the collision and return.

Replacing T in Eq. (1.12) with its equivalent in terms of T_0, we have

$$V_B = \frac{Y\sqrt{1 - v^2/c^2}}{T_0}$$

From Eq. (1.11), $$V_A = \frac{Y}{T_0}$$

If we use the classical definition of momentum, $\mathbf{p} = m\mathbf{v}$, then in frame S

$$p_A = m_A V_A = m_A\left(\frac{Y}{T_0}\right)$$

$$p_B = m_B V_B = m_B\sqrt{1 - v^2/c^2}\left(\frac{Y}{T_0}\right)$$

This means that, in this frame, momentum will not be conserved if $m_A = m_B$, where m_A and m_B are the masses as measured in S. However, if

$$m_B = \frac{m_A}{\sqrt{1 - v^2/c^2}} \tag{1.14}$$

then momentum *will* be conserved.

In the collision of Fig. 1.13 both A and B are moving in both frames. Suppose now that V_A and V_B' are very small compared with v, the relative velocity of the two frames. In this case an observer in S will see B approach A with the velocity v, make a glancing collision (since $V_B' \ll v$), and then continue on. In the limit of $V_A = 0$, if m is the mass in S of A when A is at rest, then $m_A = m$. In the limit of $V_B' = 0$, if $m(v)$ is the mass in S of B, which is moving at the velocity v, then $m_B = m(v)$. Hence Eq. (1.14) becomes

$$m(v) = \frac{m}{\sqrt{1 - v^2/c^2}} \tag{1.15}$$

We can see that if linear momentum is defined as

Relativistic momentum

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{1.16}$$

then conservation of momentum is valid in special relativity. When $v \ll c$, Eq. (1.16) becomes just $\mathbf{p} = m\mathbf{v}$, the classical momentum, as required. Equation (1.16) is often written as

Relativistic momentum

$$\mathbf{p} = \gamma m\mathbf{v} \tag{1.17}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1.18}$$

In this definition, m is the **proper mass** (or **rest mass**) of an object, its mass when measured at rest relative to an observer. (The symbol γ is the Greek letter gamma.)

"Relativistic Mass"

We could alternatively regard the increase in an object's momentum over the classical value as being due to an increase in the object's mass. Then we would call $m_0 = m$ the rest mass of the object and $m = m(v)$ from Eq. (1.17) its relativistic mass, its mass when moving relative to an observer, so that $\mathbf{p} = m\mathbf{v}$. This is the view often taken in the past, at one time even by Einstein. However, as Einstein later wrote, the idea of relativistic mass is "not good" because "no clear definition can be given. It is better to introduce no other mass concept than the 'rest mass' m." In this book the term mass and the symbol m will always refer to proper (or rest) mass, which will be considered relativistically invariant.

Figure 1.14 shows how p varies with v/c for both γmv and mv. When v/c is small, mv and γmv are very nearly the same. (For $v = 0.01c$, the difference is only 0.005 percent; for $v = 0.1c$, it is 0.5 percent, still small). As v approaches c, however, the curve for γmv rises more and more steeply (for $v = 0.9c$, the difference is 229 percent). If $v = c$, $p = \gamma mv = \infty$, which is impossible. We conclude that no material object can travel as fast as light.

But what if a spacecraft moving at $v_1 = 0.5c$ relative to the earth fires a projectile at $v_2 = 0.5c$ in the same direction? We on earth might expect to observe the projectile's speed as $v_1 + v_2 = c$. Actually, as discussed in Appendix I to this chapter, velocity addition in relativity is not so simple a process, and we would find the projectile's speed to be only $0.8c$ in such a case.

Relativistic Second Law

In relativity Newton's second law of motion is given by

Relativistic second law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{v}) \tag{1.19}$$

This is more complicated than the classical formula $\mathbf{F} = m\mathbf{a}$ because γ is a function of v. When $v \ll c$, γ is very nearly equal to 1, and $\mathbf{F}$ is very nearly equal to $m\mathbf{v}$, as it should be.

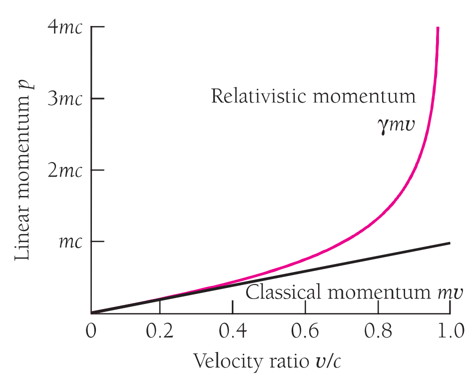


Figure 1.14 The momentum of an object moving at the velocity v relative to an observer. The mass m of the object is its value when it is at rest relative to the observer. The object's velocity can never reach c because its momentum would then be infinite, which is impossible. The relativistic momentum γmv is always correct; the classical momentum mv is valid for velocities much smaller than c.

Example 1.5

Find the acceleration of a particle of mass m and velocity $\mathbf{v}$ when it is acted upon by the constant force $\mathbf{F}$, where $\mathbf{F}$ is parallel to $\mathbf{v}$.

Solution

From Eq. (1.19), since $a = dv/dt$,

$$F = \frac{d}{dt}(\gamma m v) = m\frac{d}{dt}\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right)$$

$$= m\left[\frac{1}{\sqrt{1 - v^2/c^2}} + \frac{v^2/c^2}{(1 - v^2/c^2)^{3/2}}\right]\frac{dv}{dt}$$

$$= \frac{ma}{(1 - v^2/c^2)^{3/2}}$$

We note that F is equal to $\gamma^3 ma$, *not* to γma. Merely replacing m by γm in classical formulas does not always give a relativistically correct result.

The acceleration of the particle is therefore

$$a = \frac{F}{m}(1 - v^2/c^2)^{3/2}$$

Even though the force is constant, the acceleration of the particle decreases as its velocity increases. As $v \rightarrow c$, $a \rightarrow 0$, so the particle can never reach the speed of light, a conclusion we expect.

1.8 MASS AND ENERGY

Where $E_0 = mc^2$ comes from

The most famous relationship Einstein obtained from the postulates of special relativity—how powerful they turn out to be!—concerns mass and energy. Let us see how this relationship can be derived from what we already know.

As we recall from elementary physics, the work W done on an object by a constant force of magnitude F that acts through the distance s, where $\mathbf{F}$ is in the same direction as $\mathbf{s}$, is given by $W = Fs$. If no other forces act on the object and the object starts from rest, all the work done on it becomes kinetic energy KE, so KE $= Fs$. In the general case where F need not be constant, the formula for kinetic energy is the integral

$$\text{KE} = \int_0^s F\, ds$$

In nonrelativistic physics, the kinetic energy of an object of mass m and speed v is KE $= \frac{1}{2}mv^2$. To find the correct relativistic formula for KE we start from the relativistic form of the second law of motion, Eq. (1.19), which gives

$$\text{KE} = \int_0^s \frac{d(\gamma m v)}{dt} ds = \int_0^{mv} v\, d(\gamma m v) = \int_0^v v\, d\left(\frac{mv}{\sqrt{1 - v^2/c^2}}\right)$$

Integrating by parts ($\int x\,dy = xy - \int y\,dx$),

$$\begin{aligned}\text{KE} &= \frac{mv^2}{\sqrt{1 - v^2/c^2}} - m\int_0^v \frac{v\,dv}{\sqrt{1 - v^2/c^2}} \\ &= \frac{mv^2}{\sqrt{1 - v^2/c^2}} + \left[mc^2\sqrt{1 - v^2/c^2}\right]_0^v \\ &= \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2\end{aligned}$$

Kinetic energy

$$\text{KE} = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2 \tag{1.20}$$

This result states that the kinetic energy of an object is equal to the difference between γmc^2 and mc^2. Equation (1.20) may be written

Total energy

$$E = \gamma mc^2 = mc^2 + \text{KE} \tag{1.21}$$

If we interpret γmc^2 as the **total energy** E of the object, we see that when it is at rest and KE = 0, it nevertheless possesses the energy mc^2. Accordingly mc^2 is called the **rest energy** E_0 of something whose mass is m. We therefore have

$$E = E_0 + \text{KE}$$

where

Rest energy

$$E_0 = mc^2 \tag{1.22}$$

If the object is moving, its total energy is

Total energy

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{1.23}$$

Example 1.6

A stationary body explodes into two fragments each of mass 1.0 kg that move apart at speeds of 0.6c relative to the original body. Find the mass of the original body.

Solution

The rest energy of the original body must equal the sum of the total energies of the fragments. Hence

$$E_0 = mc^2 = \gamma m_1 c^2 + \gamma m_2 c^2 = \frac{m_1 c^2}{\sqrt{1 - v_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1 - v_2^2/c^2}}$$

and

$$m = \frac{E_0}{c^2} = \frac{(2)(1.0\text{ kg})}{\sqrt{1 - (0.60)^2}} = 2.5\text{ kg}$$

Since mass and energy are not independent entities, their separate conservation principles are properly a single one—the principle of conservation of mass energy. Mass *can* be created or destroyed, but when this happens, an equivalent amount of energy simultaneously vanishes or comes into being, and vice versa. Mass and energy are different aspects of the same thing.

It is worth emphasizing the difference between a *conserved* quantity, such as total energy, and an *invariant* quantity, such as proper mass. Conservation of E means that, in a given reference frame, the total energy of some isolated system remains the same regardless of what events occur in the system. However, the total energy may be different as measured from another frame. On the other hand, the invariance of m means that m has the same value in all inertial frames.

The conversion factor between the unit of mass (the kilogram, kg) and the unit of energy (the joule, J) is c^2, so 1 kg of matter—the mass of this book is about that—has an energy content of $mc^2 = (1\text{ kg})(3 \times 10^8\text{ m/s})^2 = 9 \times 10^{16}$ J. This is enough to send a payload of a million tons to the moon. How is it possible for so much energy to be bottled up in even a modest amount of matter without anybody having been aware of it until Einstein's work?

In fact, processes in which rest energy is liberated are very familiar. It is simply that we do not usually think of them in such terms. In every chemical reaction that evolves energy, a certain amount of matter disappears, but the lost mass is so small a fraction of the total mass of the reacting substances that it is imperceptible. Hence the "law" of conservation of mass in chemistry. For instance, only about 6×10^{-11} kg of matter vanishes when 1 kg of dynamite explodes, which is impossible to measure directly, but the more than 5 million joules of energy that is released is hard to avoid noticing.

Example 1.7

Solar energy reaches the earth at the rate of about 1.4 kW per square meter of surface perpendicular to the direction of the sun (Fig. 1.15). By how much does the mass of the sun decrease per second owing to this energy loss? The mean radius of the earth's orbit is 1.5×10^{11} m.

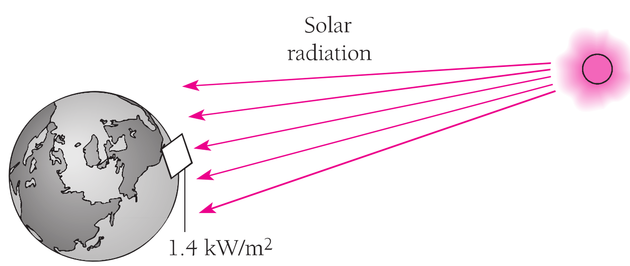


Figure 1.15

Solution

The surface area of a sphere of radius r is $A = 4\pi r^2$. The total power radiated by the sun, which is equal to the power received by a sphere whose radius is that of the earth's orbit, is therefore

$$P = \frac{P}{A}A = \frac{P}{A}(4\pi r^2) = (1.4 \times 10^3\text{ W/m}^2)(4\pi)(1.5 \times 10^{11}\text{ m})^2 = 4.0 \times 10^{26}\text{ W}$$

Thus the sun loses $E_0 = 4.0 \times 10^{26}$ J of rest energy per second, which means that the sun's rest mass decreases by

$$m = \frac{E_0}{c^2} = \frac{4.0 \times 10^{26}\text{ J}}{(3.0 \times 10^8\text{ m/s})^2} = 4.4 \times 10^9\text{ kg}$$

per second. Since the sun's mass is 2.0×10^{30} kg, it is in no immediate danger of running out of matter. The chief energy-producing process in the sun and most other stars is the conversion of hydrogen to helium in its interior. The formation of each helium nucleus is accompanied by the release of 4.0×10^{-11} J of energy, so 10^{37} helium nuclei are produced in the sun per second.

Kinetic Energy at Low Speeds

When the relative speed v is small compared with c, the formula for kinetic energy must reduce to the familiar $\frac{1}{2}mv^2$, which has been verified by experiment at such speeds. Let us see if this is true. The relativistic formula for kinetic energy is

Kinetic energy

$$\mathrm{KE} = \gamma mc^2 - mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \tag{1.20}$$

Since $v^2/c^2 \ll 1$, we can use the binomial approximation $(1 + x)^n \approx 1 + nx$, valid for $|x| \ll 1$, to obtain

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \qquad v \ll c$$

Thus we have the result

$$\mathrm{KE} \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)mc^2 - mc^2 \approx \frac{1}{2}mv^2 \qquad v \ll c$$

At low speeds the relativistic expression for the kinetic energy of a moving object does indeed reduce to the classical one. So far as is known, the correct formulation of mechanics has its basis in relativity, with classical mechanics representing an approximation that is valid only when $v \ll c$. Figure 1.16 shows how the kinetic energy of

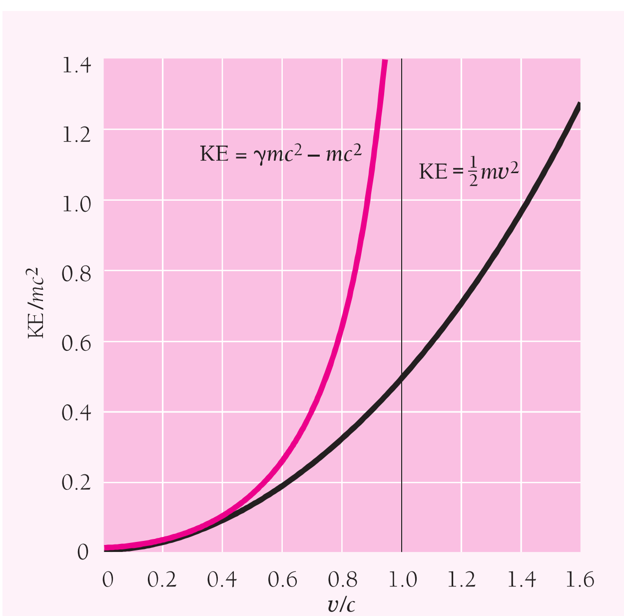


Figure 1.16 A comparison between the classical and relativistic formulas for the ratio between kinetic energy KE of a moving body and its rest energy mc^2. At low speeds the two formulas give the same result, but they diverge at speeds approaching that of light. According to relativistic mechanics, a body would need an infinite kinetic energy to travel with the speed of light, whereas in classical mechanics it would need only a kinetic energy of half its rest energy to have this speed.

a moving object varies with its speed according to both classical and relativistic mechanics.

The degree of accuracy required is what determines whether it is more appropriate to use the classical or to use the relativistic formulas for kinetic energy. For instance, when $v = 10^7$ m/s (0.033c), the formula $\frac{1}{2}mv^2$ understates the true kinetic energy by only 0.08 percent; when $v = 3 \times 10^7$ m/s (0.1c), it understates the true kinetic energy by 0.8 percent; but when $v = 1.5 \times 10^8$ m/s (0.5c), the understatement is a significant 19 percent; and when $v = 0.999c$, the understatement is a whopping 4300 percent. Since 10^7 m/s is about 6310 mi/s, the nonrelativistic formula $\frac{1}{2}mv^2$ is entirely satisfactory for finding the kinetic energies of ordinary objects, and it fails only at the extremely high speeds reached by elementary particles under certain circumstances.

1.9 ENERGY AND MOMENTUM

How they fit together in relativity

Total energy and momentum are conserved in an isolated system, and the rest energy of a particle is invariant. Hence these quantities are in some sense more fundamental than velocity or kinetic energy, which are neither. Let us look into how the total energy, rest energy, and momentum of a particle are related.

We begin with Eq. (1.23) for total energy,

Total energy

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{1.23}$$

and square it to give

$$E^2 = \frac{m^2c^4}{1 - v^2/c^2}$$

From Eq. (1.17) for momentum,

Momentum

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} \tag{1.17}$$

we find that

$$p^2c^2 = \frac{m^2v^2c^2}{1 - v^2/c^2}$$

Now we subtract p^2c^2 from E^2:

$$E^2 - p^2c^2 = \frac{m^2c^4 - m^2v^2c^2}{1 - v^2/c^2} = \frac{m^2c^4(1 - v^2/c^2)}{1 - v^2/c^2}$$

$$= (mc^2)^2$$

Hence

Energy and momentum

$$E^2 = (mc^2)^2 + p^2c^2 \tag{1.24}$$

which is the formula we want. We note that, because mc^2 is invariant, so is $E^2 - p^2c^2$: this quantity for a particle has the same value in all frames of reference.

For a system of particles rather than a single particle, Eq. (1.24) holds provided that the rest energy mc^2—and hence mass m—is that of the entire system. If the particles in the system are moving with respect to one another, the sum of their individual rest energies may not equal the rest energy of the system. We saw this in Example 1.7 when a stationary body of mass 2.5 kg exploded into two smaller bodies, each of mass 1.0 kg, that then moved apart. If we were inside the system, we would interpret the difference of 0.5 kg of mass as representing its conversion into kinetic energy of the smaller bodies. But seen as a whole, the system is at rest both before and after the explosion, so the *system* did not gain kinetic energy. Therefore the rest energy of the system includes the kinetic energies of its internal motions and it corresponds to a mass of 2.5 kg both before and after the explosion.

In a given situation, the rest energy of an isolated system may be greater than, the same as, or less than the sum of the rest energies of its members. An important case in which the system rest energy is less than the rest energies of its members is that of a system of particles held together by attractive forces, such as the neutrons and protons in an atomic nucleus. The rest energy of a nucleus (except that of ordinary hydrogen, which is a single proton) is less than the total of the rest energies of its constituent particles. The difference is called the *binding energy* of the nucleus. To break a nucleus up completely calls for an amount of energy at least equal to its binding energy. This topic will be explored in detail in Sec. 11.4. For the moment it is interesting to note how large nuclear binding energies are—nearly 10^{12} kJ per kg of nuclear matter is typical. By comparison, the binding energy of water molecules in liquid water is only 2260 kJ/kg; this is the energy needed to turn 1 kg of water at 100°C to steam at the same temperature.

Massless Particles

Can a massless particle exist? To be more precise, can a particle exist which has no rest mass but which nevertheless exhibits such particlelike properties as energy and momentum? In classical mechanics, a particle must have rest mass in order to have energy and momentum, but in relativistic mechanics this requirement does not hold.

From Eqs. (1.17) and (1.23), when $m = 0$ and $\boldsymbol{v} \ll c$, it is clear that $E = p = 0$. A massless particle with a speed less than that of light can have neither energy nor momentum. However, when $m = 0$ and $\boldsymbol{v} = c$, $E = 0/0$ and $p = 0/0$, which are indeterminate: E and p can have any values. Thus Eqs. (1.17) and (1.23) are consistent with the existence of massless particles that possess energy and momentum *provided that they travel with the speed of light.*

Equation (1.24) gives us the relationship between E and p for a particle with $m = 0$:

Massless particle

$$E = pc \tag{1.25}$$

The conclusion is not that massless particles necessarily occur, only that the laws of physics do not exclude the possibility as long as $\boldsymbol{v} = c$ and $E = pc$ for them. In fact, a massless particle—the photon—indeed exists and its behavior is as expected, as we shall find in Chap. 2.

Electronvolts

In atomic physics the usual unit of energy is the **electronvolt** (eV), where 1 eV is the energy gained by an electron accelerated through a potential difference of 1 volt. Since $W = QV$,

$$1 \text{ eV} = (1.602 \times 10^{-19} \text{ C})(1.000 \text{ V}) = 1.602 \times 10^{-19} \text{ J}$$

Two quantities normally expressed in electronvolts are the ionization energy of an atom (the work needed to remove one of its electrons) and the binding energy of a molecule (the energy needed to break it apart into separate atoms). Thus the ionization energy of nitrogen is 14.5 eV and the binding energy of the hydrogen molecule H_2 is 4.5 eV. Higher energies in the atomic realm are expressed in **kiloelectronvolts** (keV), where 1 keV $= 10^3$ eV.

In nuclear and elementary-particle physics even the keV is too small a unit in most cases, and the **megaelectronvolt** (MeV) and **gigaelectronvolt** (GeV) are more appropriate, where

$$1 \text{ MeV} = 10^6 \text{ eV} \qquad 1 \text{ GeV} = 10^9 \text{ eV}$$

An example of a quantity expressed in MeV is the energy liberated when the nucleus of a certain type of uranium atom splits into two parts. Each such fission event releases about 200 MeV; this is the process that powers nuclear reactors and weapons.

The rest energies of elementary particles are often expressed in MeV and GeV and the corresponding rest masses in MeV/c^2 and GeV/c^2. The advantage of the latter units is that the rest energy equivalent to a rest mass of, say, 0.938 GeV/c^2 (the rest mass of the proton) is just $E_0 = mc^2 = 0.938$ GeV. If the proton's kinetic energy is 5.000 GeV, finding its total energy is simple:

$$E = E_0 + \text{KE} = (0.938 + 5.000) \text{ GeV} = 5.938 \text{ GeV}$$

In a similar way the MeV/c and GeV/c are sometimes convenient units of linear momentum. Suppose we want to know the momentum of a proton whose speed is 0.800c. From Eq. (1.17) we have

$$p = \frac{m\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2/c^2}} = \frac{(0.938 \text{ GeV}/c^2)(0.800c)}{\sqrt{1 - (0.800c)^2/c^2}}$$

$$= \frac{0.750 \text{ GeV}/c}{0.600} = 1.25 \text{ GeV}/c$$

Example 1.8

An electron ($m = 0.511$ MeV/c^2) and a photon ($m = 0$) both have momenta of 2.000 MeV/c. Find the total energy of each.

Solution

(*a*) From Eq. (1.24) the electron's total energy is

$$E = \sqrt{m^2c^4 + p^2c^2} = \sqrt{(0.511 \text{ MeV}/c^2)^2c^4 + (2.000 \text{ MeV}/c)^2c^2}$$
$$= \sqrt{(0.511 \text{ MeV})^2 + (2.000 \text{ MeV})^2} = 2.064 \text{ MeV}$$

(*b*) From Eq. (1.25) the photon's total energy is

$$E = pc = (2.000 \text{ MeV}/c)c = 2.000 \text{ MeV}$$

1.10 GENERAL RELATIVITY

Gravity is a warping of spacetime

Special relativity is concerned only with inertial frames of reference, that is, frames that are not accelerated. Einstein's 1916 **general theory of relativity** goes further by including the effects of accelerations on what we observe. Its essential conclusion is that the force of gravity arises from a warping of spacetime around a body of matter (Fig. 1.17). As a result, an object moving through such a region of space in general follows a curved path rather than a straight one, and may even be trapped there.

The **principle of equivalence** is central to general relativity:

> An observer in a closed laboratory cannot distinguish between the effects produced by a gravitational field and those produced by an acceleration of the laboratory.

This principle follows from the experimental observation (to better than 1 part in 10^{12}) that the inertial mass of an object, which governs the object's acceleration when a force acts on it, is always equal to its gravitational mass, which governs the gravitational force another object exerts on it. (The two masses are actually proportional; the constant of proportionality is set equal to 1 by an appropriate choice of the constant of gravitation G.)

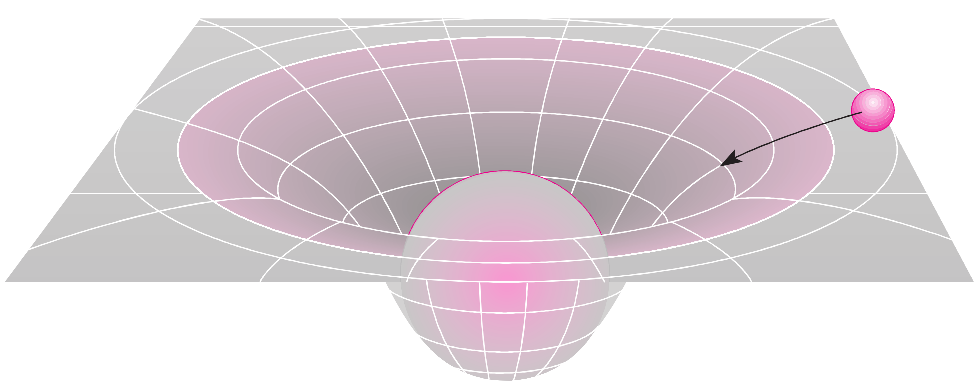


Figure 1.17 General relativity pictures gravity as a warping of spacetime due to the presence of a body of matter. An object nearby experiences an attractive force as a result of this distortion, much as a marble rolls toward the bottom of a depression in a rubber sheet. To paraphrase J. A. Wheeler, spacetime tells mass how to move, and mass tells spacetime how to curve.

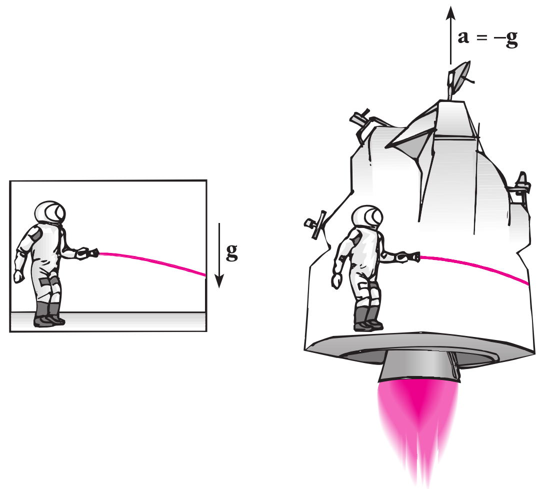


Figure 1.18 According to the principle of equivalence, events that take place in an accelerated laboratory cannot be distinguished from those which take place in a gravitational field. Hence the deflection of a light beam relative to an observer in an accelerated laboratory means that light must be similarly deflected in a gravitational field.

Gravity and Light

It follows from the principle of equivalence that light should be subject to gravity. If a light beam is directed across an accelerated laboratory, as in Fig. 1.18, its path relative to the laboratory will be curved. This means that, if the light beam is subject to the gravitational field to which the laboratory's acceleration is equivalent, the beam would follow the same curved path.

According to general relativity, light rays that graze the sun should have their paths bent toward it by 0.005°—the diameter of a dime seen from a mile away. This prediction was first confirmed in 1919 by photographs of stars that appeared in the sky near the sun during an eclipse, when they could be seen because the sun's disk was covered by the moon. The photographs were then compared with other photographs of the same part of the sky taken when the sun was in a distant part of the sky (Fig. 1.19). Einstein became a world celebrity as a result.

Because light is deflected in a gravitational field, a dense concentration of mass—such as a galaxy of stars—can act as a lens to produce multiple images of a distant light source located behind it (Fig. 1.20). A **quasar**, the nucleus of a young galaxy, is brighter than 100 billion stars but is no larger than the solar system. The first observation of gravitational lensing was the discovery in 1979 of what seemed to be a pair of nearby quasars but was actually a single one whose light was deviated by an intervening massive object. Since then a number of other gravitational lenses have been found; the effect occurs in radio waves from distant sources as well as in light waves.

The interaction between gravity and light also gives rise to the gravitational red shift and to black holes, topics that are considered in Chap. 2.

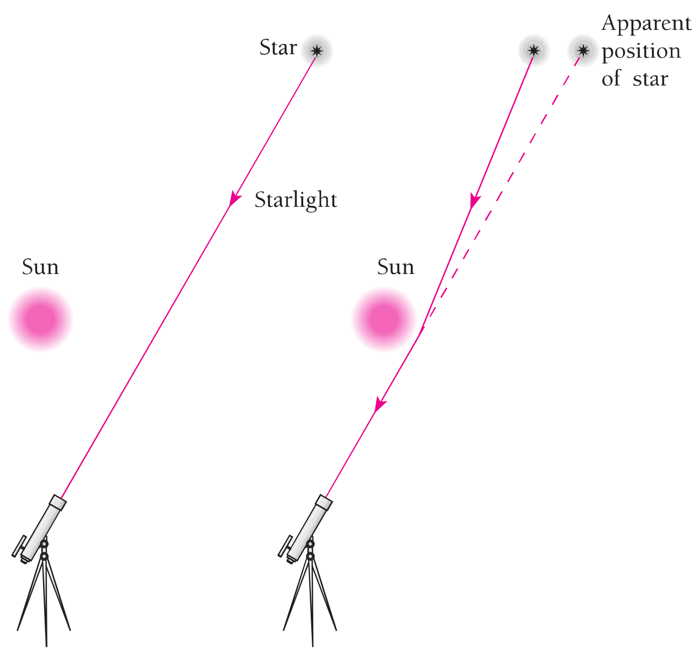


Figure 1.19 Starlight passing near the sun is deflected by its strong gravitational field. The deflection can be measured during a solar eclipse when the sun's disk is obscured by the moon.

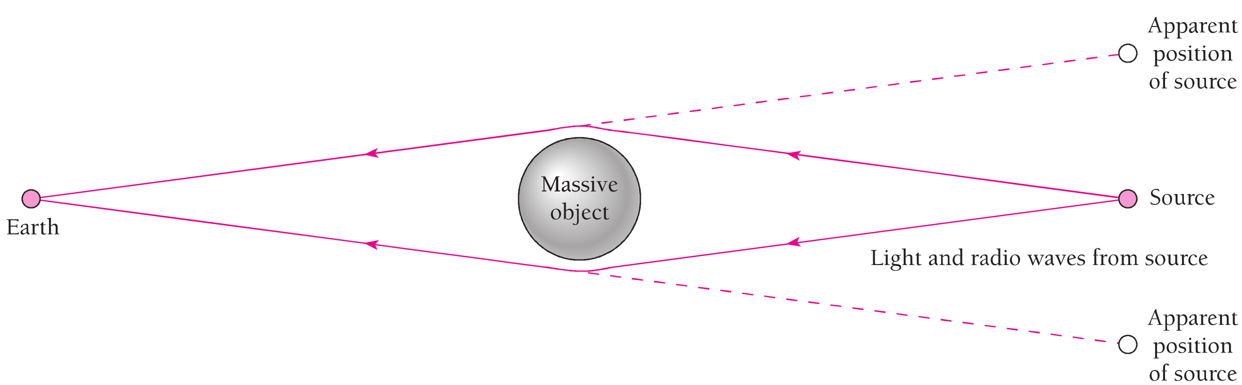


Figure 1.20 A gravitational lens. Light and radio waves from a source such as a quasar are deviated by a massive object such as a galaxy so that they seem to come from two or more identical sources. A number of such gravitational lenses have been identified.

Other Findings of General Relativity

A further success of general relativity was the clearing up of a long-standing puzzle in astronomy. The perihelion of a planetary orbit is the point in the orbit nearest the sun. Mercury's orbit has the peculiarity that its perihelion shifts (precesses) about 1.6° per century (Fig. 1.21). All but 43″ ($1'' = 1$ arc second $= \frac{1}{3600}$ of a degree) of this shift is due to the attractions of other planets, and for a while the discrepancy was used as evidence for an undiscovered planet called Vulcan whose orbit was supposed to lie

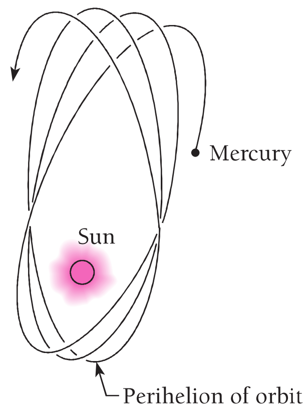


Figure 1.21 The precession of the perihelion of Mercury's orbit.

inside that of Mercury. When gravity is weak, general relativity gives very nearly the same results as Newton's formula $F = Gm_1m_2/r^2$. But Mercury is close to the sun and so moves in a strong gravitational field, and Einstein was able to show from general relativity that a precession of 43″ per century was to be expected for its orbit.

The existence of **gravitational waves** that travel with the speed of light was the prediction of general relativity that had to wait the longest to be verified. To visualize gravitational waves, we can think in terms of the model of Fig. 1.17 in which two-dimensional space is represented by a rubber sheet distorted by masses embedded in it. If one of the masses vibrates, waves will be sent out in the sheet that set other masses in vibration. A vibrating electric charge similarly sends out electromagnetic waves that excite vibrations in other charges.

A big difference between the two kinds of waves is that gravitational waves are extremely weak, so that despite much effort none have as yet been directly detected. However, in 1974 strong evidence for gravitational waves was found in the behavior of a system of two nearby stars, one a pulsar, that revolve around each other. A **pulsar** is a very small, dense star, composed mainly of neutrons, that spins rapidly and sends out flashes of light and radio waves at a regular rate, much as the rotating beam of a lighthouse does (see Sec. 9.11). The pulsar in this particular binary system emits pulses every 59 milliseconds (ms), and it and its companion (probably another neutron star) have an orbital period of about 8 h. According to general relativity, such a system should give off gravitational waves and lose energy as a result, which would reduce the orbital period as the stars spiral in toward each other. A change in orbital period means a change in the arrival times of the pulsar's flashes, and in the case of the observed binary system the orbital period was found to be decreasing at 75 ms per year. This is so close to the figure that general relativity predicts for the system that there seems to be no doubt that gravitational radiation is responsible. The 1993 Nobel Prize in physics was awarded to Joseph Taylor and Russell Hulse for this work.

Much more powerful sources of gravitational waves ought to be such events as two black holes colliding and supernova explosions in which the remnant star cores collapse into neutron stars (again, see Sec. 9.11). A gravitational wave that passes through a body of matter will cause distortions to ripple through it due to fluctuations in the gravitational field. Because gravitational forces are feeble—the electric attraction between a proton and an electron is over 10^{39} times greater than the gravitational attraction between them—such distortions at the earth induced by gravitational waves from a supernova in our galaxy (which occurs an average of once every 30 years or so) would amount to only about 1 part in 10^{18}, even less for a more distant supernova. This corresponds to a change in, say, the height of a person by well under the diameter of an atomic nucleus, yet it seems to be detectable—just—with current technology.

In one method, a large metal bar cooled to a low temperature to minimize the random thermal motions of its atoms is monitored by sensors for vibrations due to gravitational waves. In another method, an interferometer similar to the one shown in Fig. 1.2 with a laser as the light source is used to look for changes in the lengths of the arms to which the mirrors are attached. Instruments of both kinds are operating, thus far with no success.

A really ambitious scheme has been proposed that would use six spacecraft in orbit around the sun placed in pairs at the corners of a triangle whose sides are 5 million kilometers (km) long. Lasers, mirrors, and sensors in the spacecraft would detect changes in their spacings resulting from the passing of a gravitational wave. It may only be a matter of time before gravitational waves will be providing information about a variety of cosmic disturbances on the largest scale.

Appendix I to Chapter 1

The Lorentz Transformation

Suppose we are in an inertial frame of reference S and find the coordinates of some event that occurs at the time t are x, y, z. An observer located in a different inertial frame S' which is moving with respect to S at the constant velocity $\mathbf{v}$ will find that the same event occurs at the time t' and has the coordinates x', y', z'. (In order to simplify our work, we shall assume that $\mathbf{v}$ is in the $+x$ direction, as in Fig. 1.22.) How are the measurements x, y, z, t related to x', y', z', t'?

Galilean Transformation

Before special relativity, transforming measurements from one inertial system to another seemed obvious. If clocks in both systems are started when the origins of S and S' coincide, measurements in the x direction made is S will be greater than those made in S' by the amount vt, which is the distance S' has moved in the x direction. That is,

$$x' = x - vt \tag{1.26}$$

There is no relative motion in the y and z directions, and so

$$y' = y \tag{1.27}$$

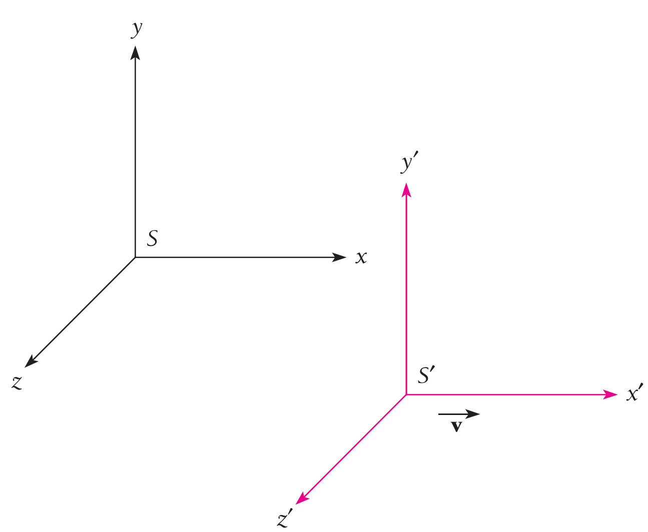


Figure 1.22 Frame S' moves in the $+x$ direction with the speed v relative to frame S. The Lorentz transformation must be used to convert measurements made in one of these frames to their equivalents in the other.

$$z' = z \tag{1.28}$$

In the absence of any indication to the contrary in our everyday experience, we further assume that

$$t' = t \tag{1.29}$$

The set of Eqs. (1.26) to (1.29) is known as the **Galilean transformation.**

To convert velocity components measured in the S frame to their equivalents in the S' frame according to the Galilean transformation, we simply differentiate x', y', and z' with respect to time:

$$v'_x = \frac{dx'}{dt'} = v_x - v \tag{1.30}$$

$$v'_y = \frac{dy'}{dt'} = v_y \tag{1.31}$$

$$v'_z = \frac{dz'}{dt'} = v_z \tag{1.32}$$

Although the Galilean transformation and the corresponding velocity transformation seem straightforward enough, they violate both of the postulates of special relativity. The first postulate calls for the same equations of physics in both the S and S' inertial frames, but the equations of electricity and magnetism become very different when the Galilean transformation is used to convert quantities measured in one frame into their equivalents in the other. The second postulate calls for the same value of the speed of light c whether determined in S or S'. If we measure the speed of light in the x direction in the S system to be c, however, in the S' system it will be

$$c' = c - v$$

according to Eq. (1.30). Clearly a different transformation is required if the postulates of special relativity are to be satisfied. We would expect both time dilation and length contraction to follow naturally from this new transformation.

Lorentz Transformation

A reasonable guess about the nature of the correct relationship between x and x' is

$$x' = k(x - vt) \tag{1.33}$$

Here k is a factor that does not depend upon either x or t but may be a function of v. The choice of Eq. (1.33) follows from several considerations:

1 It is linear in x and x', so that a single event in frame S corresponds to a single event in frame S', as it must.
2 It is simple, and a simple solution to a problem should always be explored first.
3 It has the possibility of reducing to Eq. (1.26), which we know to be correct in ordinary mechanics.

Because the equations of physics must have the same form in both S and S', we need only change the sign of $\boldsymbol{v}$ (in order to take into account the difference in the direction of relative motion) to write the corresponding equation for x in terms of x' and t':

$$x = k(x' + \boldsymbol{v}t') \tag{1.34}$$

The factor k must be the same in both frames of reference since there is no difference between S and S' other than in the sign of $\boldsymbol{v}$.

As in the case of the Galilean transformation, there is nothing to indicate that there might be differences between the corresponding coordinates y, y' and z, z' which are perpendicular to the direction of $\boldsymbol{v}$. Hence we again take

$$y' = y \tag{1.35}$$

$$z' = z \tag{1.36}$$

The time coordinates t and t', however, are *not* equal. We can see this by substituting the value of x' given by Eq. (1.33) into Eq. (1.34). This gives

$$x = k^2(x - \boldsymbol{v}t) + k\boldsymbol{v}t'$$

from which we find that

$$t' = kt + \left(\frac{1 - k^2}{k\boldsymbol{v}}\right)x \tag{1.37}$$

Equations (1.33) and (1.35) to (1.37) constitute a coordinate transformation that satisfies the first postulate of special relativity.

The second postulate of relativity gives us a way to evaluate k. At the instant $t = 0$, the origins of the two frames of reference S and S' are in the same place, according to our initial conditions, and $t' = 0$ then also. Suppose that a flare is set off at the common origin of S and S' at $t = t' = 0$, and the observers in each system measure the speed with which the flare's light spreads out. Both observers must find the same speed c (Fig. 1.23), which means that in the S frame

$$x = ct \tag{1.38}$$

and in the S' frame

$$x' = ct' \tag{1.39}$$

Substituting for x' and t' in Eq. (1.39) with the help of Eqs. (1.33) and (1.37) gives

$$k(x - \boldsymbol{v}t) = ckt + \left(\frac{1 - k^2}{k\boldsymbol{v}}\right)cx$$

and solving for x,

$$x = \frac{ckt + \boldsymbol{v}kt}{k - \left(\dfrac{1 - k^2}{k\boldsymbol{v}}\right)c} = ct\left[\frac{k + \dfrac{\boldsymbol{v}}{c}k}{k - \left(\dfrac{1 - k^2}{k\boldsymbol{v}}\right)c}\right] = ct\left[\frac{1 + \dfrac{\boldsymbol{v}}{c}}{1 - \left(\dfrac{1}{k^2} - 1\right)\dfrac{c}{\boldsymbol{v}}}\right]$$

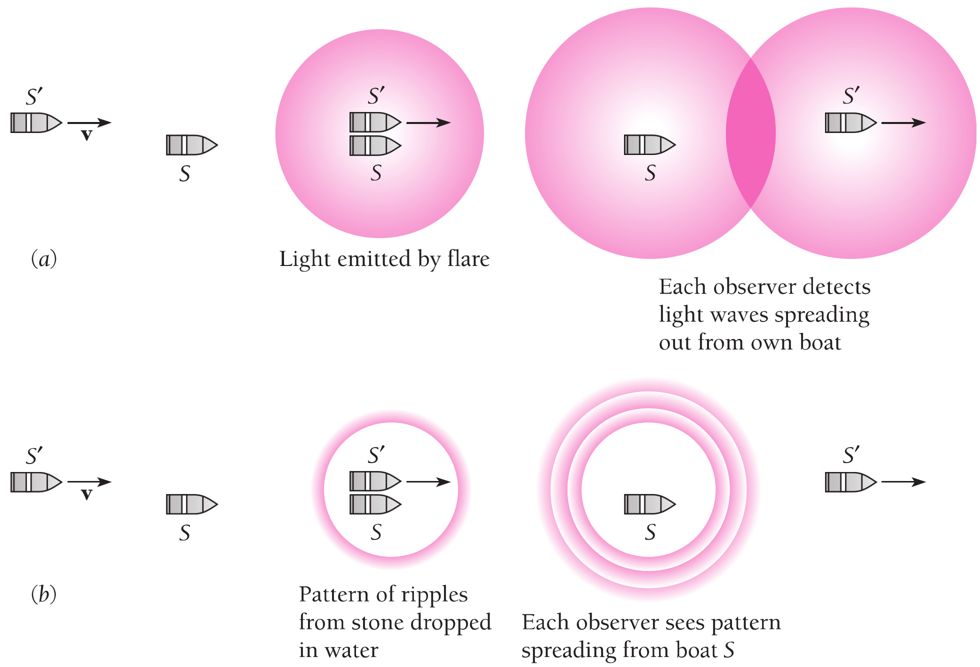


Figure 1.23 (*a*) Inertial frame S' is a boat moving at speed $\boldsymbol{v}$ in the $+x$ direction relative to another boat, which is the inertial frame S. When $t = t_0 = 0$, S' is next to S, and $x = x_0 = 0$. At this moment a flare is fired from one of the boats. An observer on boat S detects light waves spreading out at speed c from his boat. An observer on boat S' also detects light waves spreading out at speed c from her boat, even though S' is moving to the right relative to S. (*b*) If instead a stone were dropped in the water at $t = t_0 = 0$, the observers would find a pattern of ripples spreading out around S at different speeds relative to their boats. The difference between (*a*) and (*b*) is that water, in which the ripples move, is itself a frame of reference whereas space, in which light moves, is not.

This expression for x will be the same as that given by Eq. (1.38), namely, $x = ct$, provided that the quantity in the brackets equals 1. Therefore

$$\frac{1 + \dfrac{\boldsymbol{v}}{c}}{1 - \left(\dfrac{1}{k^2} - 1\right)\dfrac{c}{\boldsymbol{v}}} = 1$$

and

$$k = \frac{1}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \tag{1.40}$$

Finally we put this value of k in Eqs. (1.36) and (1.40). Now we have the complete transformation of measurements of an event made in S to the corresponding measurements made in S':

Lorentz transformation

$$x' = \frac{x - \boldsymbol{v}t}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \tag{1.41}$$

$$y' = y \tag{1.42}$$

$$z' = z \tag{1.43}$$

$$t' = \frac{t - \dfrac{vx}{c^2}}{\sqrt{1 - v^2/c^2}} \tag{1.44}$$

These equations comprise the **Lorentz transformation.** They were first obtained by the Dutch physicist H.A. Lorentz, who showed that the basic formulas of electromagnetism are the same in all inertial frames only when Eqs. (1.41) to (1.44) are used. It was not until several years later that Einstein discovered their full significance. It is obvious that the Lorentz transformation reduces to the Galilean transformation when the relative velocity v is small compared with the velocity of light c.

Example 1.9

Derive the relativistic length contraction using the Lorentz transformation.

Solution

Let us consider a rod lying along the x' axis in the moving frame S'. An observer in this frame determines the coordinates of its ends to be x'_1 and x'_2, and so the proper length of the rod is

$$L_0 = x'_2 - x'_1$$

Hendrik A. Lorentz (1853–1928) was born in Arnhem, Holland, and studied at the University of Leyden. At nineteen he returned to Arnhem and taught at the high school there while preparing a doctoral thesis that extended Maxwell's theory of electromagnetism to cover the details of the refraction and reflection of light. In 1878 he became professor of theoretical physics at Leyden, the first such post in Holland, where he remained for thirty-four years until he moved to Haarlem. Lorentz went on to reformulate and simplify Maxwell's theory and to introduce the idea that electromagnetic fields are created by electric charges on the atomic level. He proposed that the emission of light by atoms and various optical phenomena could be traced to the motions and interactions of atomic electrons. The discovery in 1896 by Pieter Zeeman, a student of his, that the spectral lines of atoms that radiate in a magnetic field are split into components of slightly different frequency confirmed Lorentz's work and led to a Nobel Prize for both of them in 1902.

The set of equations that enables electromagnetic quantities in one frame of reference to be transformed into their values in another frame of reference moving relative to the first were found by Lorentz in 1895, although their full significance was not realized until Einstein's theory of special relativity ten years afterward. Lorentz (and, independently, the Irish physicist G. F. Fitzgerald) suggested that the negative result of the Michelson-Morley experiment could be understood if lengths in the direction of motion relative to an observer were contracted. Subsequent experiments showed that although such contractions do occur, they are not the real reason for the Michelson-Morley result, which is that there is no "ether" to serve as a universal frame of reference.

In order to find $L = x_2 - x_1$, the length of the rod as measured in the stationary frame S at the time t, we make use of Eq. (1.41) to give

$$x'_1 = \frac{x_1 - \boldsymbol{v}t}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \qquad x'_2 = \frac{x_2 - \boldsymbol{v}t}{\sqrt{1 - \boldsymbol{v}^2/c^2}}$$

Hence
$$L = x_2 - x_1 = (x'_2 - x'_1)\sqrt{1 - \boldsymbol{v}^2/c^2} = L_0\sqrt{1 - \boldsymbol{v}^2/c^2}$$

This is the same as Eq. (1.9)

Inverse Lorentz Transformation

In Example 1.9 the coordinates of the ends of the moving rod were measured in the stationary frame S at the same time t, and it was easy to use Eq. (1.41) to find L in terms of L_0 and $\boldsymbol{v}$. If we want to examine time dilation, though, Eq. (1.44) is not convenient, because t_1 and t_2, the start and finish of the chosen time interval, must be measured when the moving clock is at the respective *different* positions x_1 and x_2. In situations of this kind it is easier to use the **inverse Lorentz transformation**, which converts measurements made in the moving frame S' to their equivalents in S.

To obtain the inverse transformation, primed and unprimed quantities in Eqs. (1.41) to (1.44) are exchanged, and $\boldsymbol{v}$ is replaced by $-\boldsymbol{v}$:

Inverse Lorentz transformation

$$x = \frac{x' + \boldsymbol{v}t'}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \tag{1.45}$$

$$y = y' \tag{1.46}$$

$$z' = z' \tag{1.47}$$

$$t = \frac{t' + \dfrac{\boldsymbol{v}x'}{c^2}}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \tag{1.48}$$

Example 1.10

Derive the formula for time dilation using the inverse Lorentz transformation.

Solution

Let us consider a clock at the point x' in the moving frame S'. When an observer in S' finds that the time is t'_1, an observer in S will find it to be t_1, where, from Eq. (1.48),

$$t_1 = \frac{t'_1 + \dfrac{\boldsymbol{v}x'}{c^2}}{\sqrt{1 - \boldsymbol{v}^2/c^2}}$$

After a time interval of t_0 (to him), the observer in the moving system finds that the time is now t'_2 according to his clock. That is,

$$t_0 = t'_2 - t'_1$$

The observer in S, however, measures the end of the same time interval to be

$$t_2 = \frac{t'_2 + \dfrac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

so to her the duration of the interval t is

$$t = t_2 - t_1 = \frac{t'_2 - t'_1}{\sqrt{1 - v^2/c^2}} = \frac{t_0}{\sqrt{1 - v^2/c^2}}$$

This is what we found earlier with the help of a light-pulse clock.

Velocity Addition

Special relativity postulates that the speed of light c in free space has the same value for all observers, regardless of their relative motion."Common sense" (which means here the Galilean transformation) tells us that if we throw a ball forward at 10 m/s from a car moving at 30 m/s, the ball's speed relative to the road will be 40 m/s, the sum of the two speeds. What if we switch on the car's headlights when its speed is v? The same reasoning suggests that their light, which is emitted from the reference frame S' (the car) in the direction of its motion relative to another frame S (the road), ought to have a speed of $c + v$ as measured in S. But this violates the above postulate, which has had ample experimental verification. Common sense is no more reliable as a guide in science than it is elsewhere, and we must turn to the Lorentz transformation equations for the correct scheme of velocity addition.

Suppose something is moving relative to both S and S'. An observer in S measures its three velocity components to be

$$V_x = \frac{dx}{dt} \qquad V_y = \frac{dy}{dt} \qquad V_z = \frac{dz}{dt}$$

while to an observer in S' they are

$$V'_x = \frac{dx'}{dt'} \qquad V'_y = \frac{dy'}{dt'} \qquad V'_z = \frac{dz'}{dt'}$$

By differentiating the inverse Lorentz transformation equations for x, y, z, and t, we obtain

$$dx = \frac{dx' + v\,dt'}{\sqrt{1 - v^2/c^2}} \qquad dy = dy' \qquad dz = dz' \qquad dt = \frac{dt' + \dfrac{v\,dz'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

and so $$V_x = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + \dfrac{v\,dx'}{c^2}} = \frac{\dfrac{dx'}{dt'} + v}{1 + \dfrac{v}{c^2}\dfrac{dx'}{dt'}}$$

Relativistic velocity transformation

$$V_x = \frac{V'_x + v}{1 + \dfrac{vV'_x}{c^2}} \tag{1.49}$$

Similarly,

$$V_y = \frac{V'_y\sqrt{1 - v^2/c^2}}{1 + \dfrac{vV'_x}{c^2}} \tag{1.50}$$

$$V_z = \frac{V'_z\sqrt{1 - v^2/c^2}}{1 + \dfrac{vV'_x}{c^2}} \tag{1.51}$$

If $V'_x = c$, that is, if light is emitted in the moving frame S' in its direction of motion relative to S, an observer in frame S will measure the speed

$$V_x = \frac{V'_x + v}{1 + \dfrac{vV'_x}{c^2}} = \frac{c + v}{1 + \dfrac{vc}{c^2}} = \frac{c(c + v)}{c + v} = c$$

Thus observers in the car and on the road both find the same value for the speed of light, as they must.

Example 1.11

Spacecraft Alpha is moving at $0.90c$ with respect to the earth. If spacecraft Beta is to pass Alpha at a relative speed of $0.50c$ in the same direction, what speed must Beta have with respect to the earth?

Solution

According to the Galilean transformation, Beta would need a speed relative to the earth of $0.90c + 0.50c = 1.40c$, which we know is impossible. According to Eq. (1.49), however, with $V'_x = 0.50c$ and $v = 0.90c$, the required speed is only

$$V_x = \frac{V'_x + v}{1 + \dfrac{vV'_x}{c^2}} = \frac{0.50c + 0.90c}{1 + \dfrac{(0.90c)(0.50c)}{c^2}} = 0.97c$$

which is less than c. It is necessary to go less than 10 percent faster than a spacecraft traveling at $0.90c$ in order to pass it at a relative speed of $0.50c$.

Simultaneity

The relative character of time as well as space has many implications. Notably, events that seem to take place simultaneously to one observer may not be simultaneous to another observer in relative motion, and vice versa.

Let us examine two events—the setting off of a pair of flares, say—that occur at the same time t_0 to somebody on the earth but at the different locations x_1 and x_2. What does the pilot of a spacecraft in flight see? To her, the flare at x_1 and t_0 appears at the time

$$t_1' = \frac{t_0 - \boldsymbol{v}x_1/c^2}{\sqrt{1 - \boldsymbol{v}^2/c^2}}$$

according to Eq. (1.44), while the flare at x_2 and t_0 appears at the time

$$t_2' = \frac{t_0 - \boldsymbol{v}x_2/c^2}{\sqrt{1 - \boldsymbol{v}^2/c^2}}$$

Hence two events that occur simultaneously to one observer are separated by a time interval of

$$t_2' - t_1' = \frac{\boldsymbol{v}(x_1 - x_2)/c^2}{\sqrt{1 - \boldsymbol{v}^2/c^2}}$$

to an observer moving at the speed $\boldsymbol{v}$ relative to the other observer. Who is right? The question is, of course, meaningless: both observers are "right" since each simply measures what he or she sees.

Because simultaneity is a relative concept and not an absolute one, physical theories that require simultaneity in events at different locations cannot be valid. For instance, saying that total energy is conserved in an isolated system does not rule out a process in which an amount of energy ΔE vanishes at one place while an equal amount of energy ΔE comes into being somewhere else with no actual transport of energy from one place to the other. Because simultaneity is relative, some observers of the process will find energy not being conserved. To rescue conservation of energy in the light of special relativity, then, we have to say that, when energy disappears somewhere and appears elsewhere, it has actually flowed from the first location to the second. Thus energy is conserved *locally* everywhere, not merely when an isolated system is considered—a much stronger statement of this principle.

Appendix II to Chapter 1

Spacetime

As we have seen, the concepts of space and time are inextricably mixed in nature. A length that one observer can measure with only a meter stick may have to be measured with both a meter stick and a clock by another observer.

A convenient and elegant way to express the results of special relativity is to regard events as occurring in a four-dimensional **spacetime** in which the usual three coordinates x, y, z refer to space and a fourth coordinate ict refers to time, where $i = \sqrt{-1}$. Although we cannot visualize spacetime, it is no harder to deal with mathematically than three-dimensional space.

The reason that ict is chosen as the time coordinate instead of just t is that the quantity

$$s^2 = x^2 + y^2 + z^2 - (ct)^2 \tag{1.52}$$

is **invariant** under a Lorentz transformation. That is, if an event occurs at x, y, z, t in an inertial frame S and at x', y', z', t' in another inertial frame S', then

$$s^2 = x^2 + y^2 + z^2 - (ct)^2 = x'^2 + y'^2 + z'^2 - (ct')^2$$

Because s^2 is invariant, we can think of a Lorentz transformation merely as a rotation in spacetime of the coordinate axes x, y, z, ict (Fig. 1.24).

The four coordinates x, y, z, ict define a vector in spacetime, and this *four-vector* remains fixed in spacetime regardless of any rotation of the coordinate system—that is, regardless of any shift in point of view from one inertial frame S to another S'.

Another four-vector whose magnitude remains constant under Lorentz transformations has the components p_x, p_y, p_z, iE/c. Here p_x, p_y, p_z are the usual components of the linear momentum of a body whose total energy is E. Hence the value of

$$p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c}$$

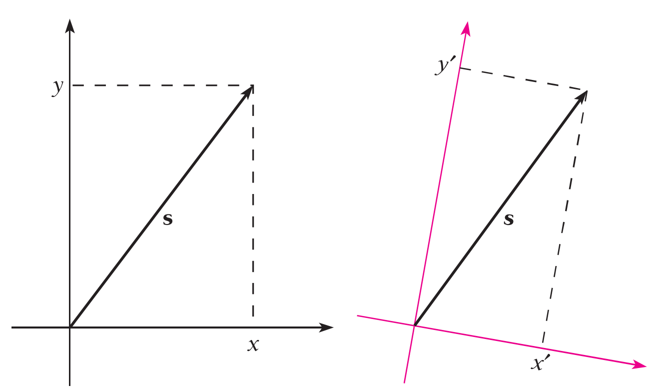


Figure 1.24 Rotating a two-dimensional coordinate system does not change the quantity $s^2 = x^2 + y^2 = x'^2 + y'^2$, where s is the length of the vector $\mathbf{s}$. This result can be generalized to the four-dimensional spacetime coordinate system x, y, z, ict.

is the same in all inertial frames even though p_x, p_y, p_z and E separately may be different. This invariance was noted earlier in connection with Eq. (1.24); we note that $p^2 = p_x^2 + p_y^2 + p_z^2$.

A more mathematically elaborate formulation brings together the electric and magnetic fields **E** and **B** into an invariant quantity called a tensor. This approach to incorporating special relativity into physics has led both to a deeper understanding of natural laws and to the discovery of new phenomena and relationships.

Spacetime Intervals

The statements made at the end of Sec. 1.2 (P. 10) are easy to confirm using the idea of spacetime. Figure 1.25 shows two events plotted on the axes x and ct. Event 1 occurs at $x = 0$, $t = 0$ and event 2 occurs at $x = \Delta x$, $t = \Delta t$. The *spacetime interval* Δs between them is defined by

Spacetime interval between events

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 \tag{1.53}$$

The virtue of this definition is that $(\Delta s)^2$, like the s^2 of Eq. 1.52, is invariant under Lorentz transformations. If Δx and Δt are the differences in space and time between two events measured in the S frame and $\Delta x'$ and $\Delta t'$ are the same quantities measured in the S' frame,

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - (\Delta x')^2$$

Therefore whatever conclusions we arrive at in the S frame in which event 1 is at the origin hold equally well in any other frame in relative motion at constant velocity.

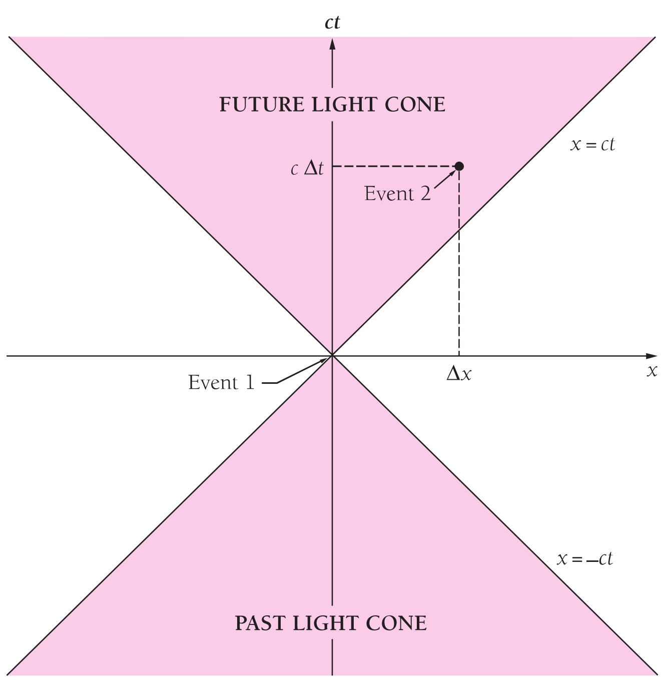


Figure 1.25 The past and future light cones in spacetime of event 1.

Now let us look into the possible relationships between events 1 and 2. Event 2 can be related causally in some way to event 1 provided that a signal traveling slower than the speed of light can connect these events, that is, provided that

$$c\Delta t > |\Delta x|$$

or

Timelike interval $$(\Delta s)^2 > 0 \tag{1.53}$$

An interval in which $(\Delta s)^2 > 0$ is said to be *timelike*. Every timelike interval that connects event 1 with another event lies within the *light cones* bounded by $x = \pm ct$ in Fig. 1.25. All events that could have affected event 1 lie in the past light cone; all events that event 1 is able to affect lie in the future light cone. (Events connected by timelike intervals need not *necessarily* be related, of course, but it is *possible* for them to be related.)

Conversely, the criterion for there being no causal relationship between events 1 and 2 is that

$$c\Delta t < |\Delta x|$$

or

Spacelike interval $$(\Delta s)^2 < 0 \tag{1.54}$$

An interval in which $(\Delta s)^2 < 0$ is said to be *spacelike*. Every event that is connected with event 1 by a spacelike interval lies outside the light cones of event 1 and neither has interacted with event 1 in the past nor is capable of interacting with it in the future; the two events must be entirely unrelated.

When events 1 and 2 can be connected with a light signal only,

$$c\Delta t = |\Delta x|$$

or

Lightlike interval $$\Delta s = 0 \tag{1.55}$$

An interval in which $\Delta s = 0$ is said to be *lightlike*. Events that can be connected with event 1 by lightlike intervals lie on the boundaries of the light cones.

These conclusions hold in terms of the light cones of event 2 because $(\Delta s)^2$ is invariant; for example, if event 2 is inside the past light cone of event 1, event 1 is inside the future light cone of event 2. In general, events that lie in the future of an event as seen in one frame of reference S lie in its future in every other frame S', and events that lie in the past of an event in S lie in its past in every other frame S'. Thus "future" and "past" have invariant meanings. However, "simultaneity" is an ambiguous concept, because all events that lie outside the past and future light cones of event 1 (that is, all events connected by spacelike intervals with event 1) can appear to occur simultaneously with event 1 in some particular frame of reference.

The path of a particle in spacetime is called its *world line* (Fig. 1.26). The world line of a particle must lie within its light cones.

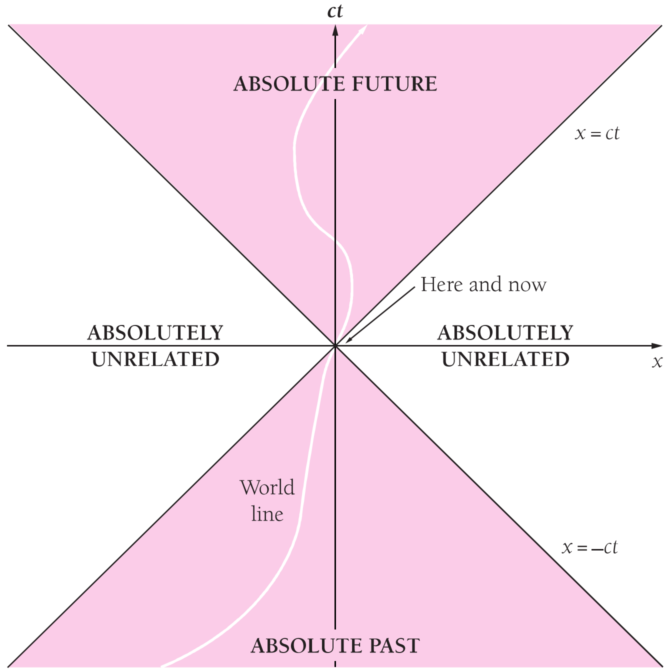


Figure 1.26 The world line of a particle in spacetime.

EXERCISES

But be ye doers of the word, and not hearers only, deceiving your own selves. —James I:22

1.1 Special Relativity

1. If the speed of light were smaller than it is, would relativistic phenomena be more or less conspicuous than they are now?

2. It is possible for the electron beam in a television picture tube to move across the screen at a speed faster than the speed of light. Why does this not contradict special relativity?

1.2 Time Dilation

3. An athlete has learned enough physics to know that if he measures from the earth a time interval on a moving spacecraft, what he finds will be greater than what somebody on the spacecraft would measure. He therefore proposes to set a world record for the 100-m dash by having his time taken by an observer on a moving spacecraft. Is this a good idea?

4. An observer on a spacecraft moving at 0.700c relative to the earth finds that a car takes 40.0 min to make a trip. How long does the trip take to the driver of the car?

5. Two observers, A on earth and B in a spacecraft whose speed is 2.00×10^8 m/s, both set their watches to the same time when the ship is abreast of the earth. (*a*) How much time must elapse by A's reckoning before the watches differ by 1.00 s? (*b*) To A, B's watch seems to run slow. To B, does A's watch seem to run fast, run slow, or keep the same time as his own watch?

6. An airplane is flying at 300 m/s (672 mi/h). How much time must elapse before a clock in the airplane and one on the ground differ by 1.00 s?

7. How fast must a spacecraft travel relative to the earth for each day on the spacecraft to correspond to 2 d on the earth?

8. The Apollo 11 spacecraft that landed on the moon in 1969 traveled there at a speed relative to the earth of 1.08×10^4 m/s. To an observer on the earth, how much longer than his own day was a day on the spacecraft?

9. A certain particle has a lifetime of 1.00×10^{-7} s when measured at rest. How far does it go before decaying if its speed is 0.99c when it is created?

1.3 Doppler Effect

10. A spacecraft receding from the earth at 0.97c transmits data at the rate of 1.00×10^4 pulses/s. At what rate are they received?

11. A galaxy in the constellation Ursa Major is receding from the earth at 15,000 km/s. If one of the characteristic wavelengths of the light the galaxy emits is 550 nm, what is the corresponding wavelength measured by astronomers on the earth?

12. The frequencies of the spectral lines in light from a distant galaxy are found to be two-thirds as great as those of the same lines in light from nearby stars. Find the recession speed of the distant galaxy.

13. A spacecraft receding from the earth emits radio waves at a constant frequency of 10^9 Hz. If the receiver on earth can measure frequencies to the nearest hertz, at what spacecraft speed can the difference between the relativistic and classical doppler effects be detected? For the classical effect, assume the earth is stationary.

14. A car moving at 150 km/h (93 mi/h) is approaching a stationary police car whose radar speed detector operates at a frequency of 15 GHz. What frequency change is found by the speed detector?

15. If the angle between the direction of motion of a light source of frequency ν_0 and the direction from it to an observer is θ, the frequency ν the observer finds is given by

$$\nu = \nu_0 \frac{\sqrt{1 - v^2/c^2}}{1 - (v/c)\cos\theta}$$

where v is the relative speed of the source. Show that this formula includes Eqs. (1.5) to (1.7) as special cases.

16. (*a*) Show that when $v \ll c$, the formulas for the doppler effect both in light and in sound for an observer approaching a source, and vice versa, all reduce to $\nu \approx \nu_0(1 + v/c)$, so that $\Delta\nu/\nu \approx v/c$. [*Hint*: For $x \ll 1$, $1/(1 + x) \approx 1 - x$.] (*b*) What do the formulas for an observer receding from a source, and vice versa, reduce to when $v \ll c$?

1.4 Length Contraction

17. An astronaut whose height on the earth is exactly 6 ft is lying parallel to the axis of a spacecraft moving at 0.90*c* relative to the earth. What is his height as measured by an observer in the same spacecraft? By an observer on the earth?

18. An astronaut is standing in a spacecraft parallel to its direction of motion. An observer on the earth finds that the spacecraft speed is 0.60*c* and the astronaut is 1.3 m tall. What is the astronaut's height as measured in the spacecraft?

19. How much time does a meter stick moving at 0.100*c* relative to an observer take to pass the observer? The meter stick is parallel to its direction of motion.

20. A meter stick moving with respect to an observer appears only 500 mm long to her. What is its relative speed? How long does it take to pass her? The meter stick is parallel to its direction of motion.

21. A spacecraft antenna is at an angle of 10° relative to the axis of the spacecraft. If the spacecraft moves away from the earth at a speed of 0.70*c*, what is the angle of the antenna as seen from the earth?

1.5 Twin Paradox

22. Twin *A* makes a round trip at 0.6*c* to a star 12 light-years away, while twin *B* stays on the earth. Each twin sends the other a signal once a year by his own reckoning. (*a*) How many signals does *A* send during the trip? How many does *B* send? (*b*) How many signals does *A* receive? How many does *B* receive?

23. A woman leaves the earth in a spacecraft that makes a round trip to the nearest star, 4 light-years distant, at a speed of 0.9*c*. How much younger is she upon her return than her twin sister who remained behind?

1.7 Relativistic Momentum

24. (*a*) An electron's speed is doubled from 0.2*c* to 0.4*c*. By what ratio does its momentum increase? (*b*) What happens to the momentum ratio when the electron's speed is doubled again from 0.4*c* to 0.8*c*?

25. All definitions are arbitrary, but some are more useful than others. What is the objection to defining linear momentum as $\mathbf{p} = m\mathbf{v}$ instead of the more complicated $\mathbf{p} = \gamma m\mathbf{v}$?

26. Verify that

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{p^2}{m^2c^2}$$

1.8 Mass and Energy

27. Dynamite liberates about 5.4×10^6 J/kg when it explodes. What fraction of its total energy content is this?

28. A certain quantity of ice at 0°C melts into water at 0°C and in so doing gains 1.00 kg of mass. What was its initial mass?

29. At what speed does the kinetic energy of a particle equal its rest energy?

30. How many joules of energy per kilogram of rest mass are needed to bring a spacecraft from rest to a speed of 0.90*c*?

31. An electron has a kinetic energy of 0.100 MeV. Find its speed according to classical and relativistic mechanics.

32. Verify that, for $E \gg E_0$,

$$\frac{v}{c} \approx 1 - \frac{1}{2}\left(\frac{E_0}{E}\right)^2$$

33. A particle has a kinetic energy 20 times its rest energy. Find the speed of the particle in terms of *c*.

34. (*a*) The speed of a proton is increased from 0.20*c* to 0.40*c*. By what factor does its kinetic energy increase? (*b*) The proton speed is again doubled, this time to 0.80*c*. By what factor does its kinetic energy increase now?

35. How much work (in MeV) must be done to increase the speed of an electron from 1.2×10^8 m/s to 2.4×10^8 m/s?

36. (*a*) Derive a formula for the minimum kinetic energy needed by a particle of rest mass *m* to emit Cerenkov radiation in a medium of index of refraction *n*. [*Hint*: Start from Eqs. (1.21) and (1.23).] (*b*) Use this formula to find KE_{min} for an electron in a medium of $n = 1.5$.

37. Prove that $\frac{1}{2}\gamma mv^2$, does *not* equal the kinetic energy of a particle moving at relativistic speeds.

38. A moving electron collides with a stationary electron and an electron-positron pair comes into being as a result (a positron is a positively charged electron). When all four particles have the same velocity after the collision, the kinetic energy required for this process is a minimum. Use a relativistic calculation to show that $KE_{min} = 6mc^2$, where *m* is the rest mass of the electron.